Three Point Functions in the Large $\mathcal{N}=4$ Holography

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Abstract

The 16 higher spin currents with spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$, $(\frac{3}{2}, 2, 2, \frac{5}{2})$, $(\frac{3}{2}, 2, 2, \frac{5}{2})$ and $(2, \frac{5}{2}, \frac{5}{2}, 3)$ in an extension of large $\mathcal{N}=4$ 'nonlinear' superconformal algebra in two dimensions were obtained previously. By analyzing the zero mode eigenvalue equations carefully, the three point functions of bosonic (higher spin) currents with two scalars for any finite N (where $SU(N+2)$ is the group of coset) and k (the level of spin-1 Kac Moody current) are obtained. Furthermore, the 16 higher spin currents with above spin contents in an extension of large $\mathcal{N}=4$ 'linear' superconformal algebra are obtained for generic N and k implicitly. The corresponding three point functions are also determined. Under the large N 't Hooft limit, the two corresponding three point functions in the nonlinear and linear versions coincide with each other although they are completely different for finite N and k.

1 Introduction

One of the consistency checks in the large $\mathcal{N}=4$ holography [1] can be described as the matching of correlation functions where one can see more dynamical information. One would like to match the two dimensional conformal field theory (CFT) answer with the predictions from the bulk Vasiliev (or its generalization) theory. The simplest three-point functions involve two scalar primaries with one higher spin current. Then one needs to obtain the eigenvalue equations for the zero modes of the higher spin currents acting on the coset scalar primaries. The main motivation of [2] was to construct the higher spin currents for general N and k in order to see these eigenvalue equations explicitly. Here the positive integer N appears in the group $G=SU(N+2)$ of the $\mathcal{N}=4$ coset theory in two dimensional CFT while the positive integer k is the level of bosonic spin-1 (affine Kac-Moody) current. See also the relevant works in [3, 4] for the similar duality in the bosonic theory and there exists a review paper [5] where one can find the relevant works in the context of higher spin AdS/CFT correspondence.

Before describing the eigenvalue equations, let us recall what is the large $\mathcal{N}=4$ coset theory in two dimensions. More explicitly, the $\mathcal{N}=4$ coset theory is described by the following 'supersymmetric' coset

$$\text{Wolf}\times SU(2)\times U(1)=\frac{SU(N+2)}{SU(N)},$$

where N is odd. The basic currents are given by the bosonic spin-1 current $V^a(z)$ and the fermionic spin-$\frac{1}{2}$ current $Q^b(z)$. The operator product expansion (OPE) between these currents does not have any singular term. The indices run over $a,b,\cdots=1,2,\cdots,\frac{(N+2)^2-1}{2}$, $1^*,2^*,\cdots,(\frac{(N+2)^2-1}{2})^*$. The number $(N+2)^2-1$ is the dimension of $G=SU(N+2)$ group. For the extension of the $\mathcal{N}=4$ 'nonlinear' superconformal algebra, the relevant coset is given by the Wolf space itself $\frac{SU(N+2)}{SU(N)\times SU(2)\times U(1)}$. For the extension of the $\mathcal{N}=4$ 'linear' superconformal algebra, the corresponding coset is given by the Wolf space multiplied by $SU(2)\times U(1)$ which is equivalent to the above coset $\frac{SU(N+2)}{SU(N)}$. In our previous work in [2], the explicit 16 lowest higher spin currents (which are multiple product of the above basic currents together with their derivatives) were expressed in terms of the Wolf space coset fields explicitly. These findings allow us to calculate the zero modes for the higher spin currents in terms of the generators of the $G=SU(N+2)$ because the zero modes of the spin-1 current satisfy the defining commutation relations of the underlying finite dimensional Lie algebra $SU(N+2)$. Furthermore, all the OPEs between the higher spin currents and the spin-$\frac{1}{2}$ current are determined explicitly by construction.

The minimal representations of [1] are given by two representations. One of the minimal representation is given by $(0;f)$ where the nonnegative integer mode of the spin-1 current $V^a(z)$ in the $SU(N+2)$ acting on the state $|(0;f)>$ vanishes. Under the decomposition of $SU(N+2)$ into the $SU(N)\times SU(2)$, the adjoint representation of $SU(N+2)$ breaks into as follows: $(\mathbf{N+2})^{\mathbf{2}}-\mathbf{1} \rightarrow (\mathbf{N^2-1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1})\oplus(\mathbf{N},\mathbf{2})\oplus(\bar{\mathbf{N}},\mathbf{2})$. Among these representations, the fundamental representation for the $SU(N)$ is given by $(\mathbf{N},\mathbf{2})$. Therefore, the representation $(0;f)$ corresponds to these representations $(\mathbf{N},\mathbf{2})$. Similarly, the representation $(0;\bar{f})$ corresponds to these representations $(\bar{\mathbf{N}},\mathbf{2})$. The corresponding states for the representation $(0;f)$ are given by $-\frac{1}{2}$ mode of the spin-$\frac{1}{2}$ current $Q^a(z)$ acting on the vacuum $|0>$ (corresponding to (4.2) of [1]) where the index a is restricted to the $2N$ coset index [1]. As described in [1], from the OPEs between the spin-1 currents $A^{+i}(z)$ and the spin-$\frac{1}{2}$ current $Q^a(w)$ with coset index, the states $|(0;f)>$ are singlets with respect to the spin-1 currents $A^{+i}(z)$. The eigenvalue for the zero mode in the (higher spin) currents (multiple product of the above spin-1 and spin-$\frac{1}{2}$ currents) acting on this state can be obtained from the highest pole of the OPE between the (higher spin) current and the spin-$\frac{1}{2}$ current [2].

Other of the minimal representation is given by $(f;0)$ where the positive half integer mode of the spin-$\frac{1}{2}$ current $Q^a(z)$ in the $SU(N+2)$ acting on the state $|(f;0)>$ vanishes. The positive integer modes of the spin-1 current should annihilate this state. They are the singlets with respect to the $SU(N)$ in the $SU(N+2)$ representation based on the fundamental representation. That is, the fundamental representation $(\mathbf{N+2})$ of $SU(N+2)$ transforms as a singlet $(\mathbf{1},\mathbf{2})_{-N}$ with respect to $SU(N)$ under the following branching $(\mathbf{N+2})\rightarrow(\mathbf{N},\mathbf{1})_2\oplus(\mathbf{1},\mathbf{2})_{-N}$ with respect to $SU(N)\times SU(2)\times U(1)$. The indices 2 and $-N$ denote the $U(1)$ charge which will be described later in (3.10) [3]. On the other hand, $(\mathbf{N},\mathbf{1})_2$ refers to the fundamental representation with respect to $SU(N)$ and describes the light state $|(f;f)>$. For the state $|(f;0)>$, the $SU(N+2)$ generator T_{a^*} corresponds to the zero mode of the spin-1 current $V^a(z)$ because the the zero mode of the spin-1 current satisfies the commutation relation of the underlying finite dimensional Lie algebra $SU(N+2)$. Then the nontrivial contributions to the

[1]One can further classify two independent states denoted by $|(0;f)>_+$ with N coset indices and $|(0;f)>_-$ with other N coset indices [6] by emphasizing that $\pm$ refer to the doublet of $SU(2)$.

[2] Furthermore, there exist the nontrivial states for the negative half integer mode (as well as the $\frac{1}{2}$ mode) of the spin-$\frac{1}{2}$ current acting on this state $|(0;f)>$ because the action of those negative mode of spin-$\frac{1}{2}$ current on the vacuum $|0>$ is nonzero. Note that the action of $\frac{1}{2}$ mode for the spin-$\frac{1}{2}$ current on the state $|(0;f)>$ can be written in terms of the anticommutator of these modes acting on the vacuum $|0>$ which is nonzero [7, 8, 9]. The positive half integer modes of the spin-$\frac{1}{2}$ current ($\frac{3}{2},\frac{5}{2},\cdots$ modes) acting on the state $|(0;f)>$ vanish.

[3]In this case also, the states are further classified as $|(f;0)>_+$ and $|(f;0)>_-$ with explicit $SU(2)$ indices. For the antifundamental representation of $SU(N+2)$, one has the following branching rule $(\overline{\mathbf{N+2}})\rightarrow(\bar{\mathbf{N}},\mathbf{1})_{-2}\oplus(\mathbf{1},\mathbf{2})_N$ with respect to $SU(N)\times SU(2)\times U(1)$.

zero mode (of (higher spin) currents) eigenvalue equation associated with the state $|(f;0)>$ come from the multiple product of the spin-1 current $V^a(z)$ in the (higher spin) currents [4]. After substituting the $SU(N+2)$ generator T_{a^*} into the zero mode of spin-1 current V_0^a in the multiple product of the (higher spin) currents, one obtains the $(N+2)\times(N+2)$ matrix acting on the state $|(f;0)>$. Then the last 2×2 subdiagonal matrix is associated with the above $SU(2)\times U(1)$ group. The eigenvalue can be read off from the last each diagonal matrix element in this 2×2 matrix. Furthermore, the first $N\times N$ subdiagonal matrix provides the corresponding eigenvalues (for the higher spin currents) for the light state $|(f;f)>$ as mentioned before.

It is known under the large level limit that the perturbative Vasiliev theory is a subsector of the tensionless string theory [10]. The corresponding CFT is based on the small $\mathcal{N}=4$ linear superconformal algebra. For finite level, the large $\mathcal{N}=4$ linear superconformal algebra plays an important role. Then it is natural to consider the extension of large $\mathcal{N}=4$ linear superconformal algebra. The coset realization for the large $\mathcal{N}=4$ linear superconformal algebra has been done by Saulina [11]. See also [12]. Compared to the spin-$\frac{3}{2}$ currents in the large $\mathcal{N}=4$ nonlinear superconformal algebra, the extra terms (which are cubic in the spin-$\frac{1}{2}$ currents) in the spin-$\frac{3}{2}$ currents of large $\mathcal{N}=4$ linear superconformal algebra occur. Moreover, the contractions between the basic currents living in the coset $\frac{SU(N+2)}{SU(N)}$ contain the extra coset indices corresponding to the above $SU(2)\times U(1)$. One can repeat the procedures described in [2] and would like to construct the higher spin currents in an extension of the large $\mathcal{N}=4$ linear superconformal algebra. In [13], the explicit coset realization for $N=3$ in the large $\mathcal{N}=4$ linear superconformal algebra has been found.

In section 2, the large $\mathcal{N}=4$ nonlinear superconformal algebra and its extension are reviewed [2, 13, 14] and the $SU(N)$ subgroup appears in the first $N\times N$ matrix inside of $(N+2)\times(N+2)$ matrix.

In section 3, based on the section 2, the eigenvalue equations of spin-2 stress energy tensor for the above two minimal states. Next, the eigenvalue equations of higher spin currents with spins-$1,2$ and 3 for the above two minimal states are presented. The corresponding three-point functions are described.

[4]When the spin-$\frac{1}{2}$ current is present in the (higher spin) current $\partial^i Q J^{(s-i-\frac{1}{2})}$ where $J^{(s-i-\frac{1}{2})}$ stands for composite field between the spin-1 currents with spin $(s-i-\frac{1}{2})$, then the zero mode contains $\sum_{p=-i+\frac{1}{2}}^{-\frac{1}{2}} J_{-p}^{(s-i-\frac{1}{2})}(\partial^i Q)_p|(f;0)>$. The group indices are ignored. From the mode expansion $Q^a(z)=\sum_{m=-\infty}^{\infty}\frac{Q_m^a}{z^{m+\frac{1}{2}}}$, the p mode of $(\partial^i Q)$ can be written as $(p+\frac{1}{2})(p+\frac{3}{2})\cdots(p+\frac{(2i-1)}{2})Q_p$ up to an overall factor. Therefore, one realizes that for each p value in the summation, the coefficient in Q_p vanishes and there is no contribution in the eigenvalue equation. There are also the terms containing $(\partial^i Q)_p J_{-p}^{(s-i-\frac{1}{2})}|(f;0)>$ with negative p which do not produce any contribution.

In section 4, by summarizing the previous work by Saulina on the large $\mathcal{N}=4$ linear superconformal algebra, its extension is described.

In section 5, based on the section 4, the eigenvalue equations of spin-2 stress energy tensor (which is different from the one in section 2) for the above two minimal states. Next, the eigenvalue equations of higher spin currents with spins-$1, 2$ and 3 for the above two minimal states are given. The corresponding three-point functions are described.

In section 6, the summary of this paper is given and the future directions are described briefly.

In Appendices A-F, some details in sections $2, 3, 4, 5$ are presented.

The Thielemans package [15] is used in this paper.

2 The large $\mathcal{N}=4$ nonlinear superconformal algebra and its extension in the Wolf space coset: review

In order to compare the previous results given in [1, 16] with our findings explicitly, one should associate the subgroup of the group in the coset theory with the first $N \times N$ matrix inside of $(N+2)\times(N+2)$ matrix. Then the corresponding coset indices occur in the remaining row and column elements except the last 2×2 matrix elements.

2.1 The $\mathcal{N}=1$ Kac-Moody current algebra in component approach

The generators of $G=SU(N+2)$ in the Wolf space coset are given in Appendix A. They satisfy the commutation relation $[T_a, T_b] = f_{ab}{}^{c} T_c$ where the indices run over $a, b, \cdots = 1, 2, \cdots, \frac{(N+2)^2-1}{2}, 1^*, 2^*, \cdots, (\frac{(N+2)^2-1}{2})^*$. The normalization is as follows: $g_{ab} = \frac{1}{2c_G} f_{ac}{}^{d} f_{bd}{}^{c}$ where c_G is the dual Coxeter number of the group G. The operator product expansions between the spin-1 and the spin-$\frac{1}{2}$ currents are described as [17] [5]

$$
\begin{aligned}
V^a(z)\, V^b(w) &= \frac{1}{(z-w)^2}\, k\, g^{ab} - \frac{1}{(z-w)}\, f^{ab}{}_{c}\, V^c(w) + \cdots, \\
Q^a(z)\, Q^b(w) &= -\frac{1}{(z-w)}\, (k+N+2)\, g^{ab} + \cdots, \\
V^a(z)\, Q^b(w) &= +\cdots.
\end{aligned}
\tag{2.1}
$$

Here k is the level and a positive integer. Note that there is no singular term in the OPE between the spin-1 current $V^a(z)$ and the spin-$\frac{1}{2}$ current $Q^b(w)$.

[5] In the work of [16], the different normalization for the spin-$\frac{1}{2}$ currents is used. The right hand side of their OPE between $\psi^{i,\alpha}(z)$ and $\bar{\psi}^{j,\beta}(w)$ has the first order pole with weight 1. Then our $\frac{1}{\sqrt{k+N+2}} Q^a(z)$ corresponds to their spin-$\frac{1}{2}$ currents.

2.2 The large $\mathcal{N}=4$ nonlinear superconformal algebra in the Wolf space coset

The Wolf space coset is given by

$$\text{Wolf} = \frac{G}{H} = \frac{SU(N+2)}{SU(N)\times SU(2)\times U(1)}. \tag{2.2}$$

The group indices are divided into

$$\begin{aligned} G \quad \text{indices} \quad &: \quad a, b, c, \cdots = 1, 2, \cdots, \frac{(N+2)^2-1}{2}, 1^*, 2^*, \cdots, (\frac{(N+2)^2-1}{2})^*, \\ \frac{G}{H} \quad \text{indices} \quad &: \quad \bar{a}, \bar{b}, \bar{c}, \cdots = 1, 2, \cdots, 2N, \cdots, 1^*, 2^*, \cdots, 2N^*. \end{aligned} \tag{2.3}$$

For given $(N+2)\times(N+2)$ matrix, one can associate the above $4N$ coset indices as follows [1]:

$$\left(\begin{array}{ccccc|cc} & & & & & * & * \\ & & & & & * & * \\ & & & & & \vdots & \vdots \\ & & & & & * & * \\ & & & & & * & * \\ \hline * & * & \cdots & * & * & & \\ * & * & \cdots & * & * & & \end{array}\right)_{(N+2)\times(N+2)} . \tag{2.4}$$

Because the different embedding is used in this paper, the different matrix representations of $h^i_{\bar{a}\bar{b}}$ and $d^0_{\bar{a}\bar{b}}$ will be given. However the large $\mathcal{N}=4$ nonlinear superconformal algebra remains unchanged although the 11 currents look differently as one expects. The relevant structure corresponding to $SU(N+2)$ appears in Appendix A where the adjoint index a is written in terms of complex basis.

Then the 11 currents of large $\mathcal{N}=4$ nonlinear superconformal algebra in terms of $\mathcal{N}=1$ Kac-Moody currents $V^a(z)$ and $Q^{\bar{b}}(z)$ together with the three almost complex structures $h^i_{\bar{a}\bar{b}} (i=1,2,3)$ are obtained. Furthermore, the three almost complex structures (h^1, h^2, h^3) are antisymmetric rank-two tensors and satisfy the algebra of imaginary quaternions [11]

$$h^i_{\bar{a}\bar{c}}\, h^{j\bar{c}}{}_{\bar{b}} \quad = \quad \epsilon^{ijk}\, h^k_{\bar{a}\bar{b}} - \delta^{ij}\, g_{\bar{a}\bar{b}}. \tag{2.5}$$

By collecting the 11 currents in [2] together, the explicit 11 currents of large $\mathcal{N}=4$ nonlinear superconformal algebra with (2.3) are given by [6]

$$G^0(z) \quad = \quad \frac{i}{(k+N+2)}\, Q_{\bar{a}}\, V^{\bar{a}}(z), \qquad G^i(z) = \frac{i}{(k+N+2)}\, h^i_{\bar{a}\bar{b}}\, Q^{\bar{a}}\, V^{\bar{b}}(z),$$

[6] One has the following relations between the spin-$\frac{3}{2}$ currents with double index notation and those with a

$$
\begin{aligned}
A^{+i}(z) &= -\frac{1}{4N}\, f^{\bar{a}\bar{b}}{}_{c}\, h^{i}_{\bar{a}\bar{b}}\, V^{c}(z), \qquad A^{-i}(z) = -\frac{1}{4(k+N+2)}\, h^{i}_{\bar{a}\bar{b}}\, Q^{\bar{a}}\, Q^{\bar{b}}(z), \\
T(z) &= \frac{1}{2(k+N+2)^2}\left[(k+N+2)\, V_{\bar{a}}\, V^{\bar{a}} + k\, Q_{\bar{a}}\, \partial\, Q^{\bar{a}} + f_{\bar{a}\bar{b}c}\, Q^{\bar{a}}\, Q^{\bar{b}}\, V^{c}\right](z) \\
& - \frac{1}{(k+N+2)}\sum_{i=1}^{3}\left(A^{+i}+A^{-i}\right)^2(z), \qquad (2.6)
\end{aligned}
$$

where $i = 1,2,3$. The $G^{\mu}(z)$ currents are four supersymmetry currents, $A^{\pm i}(z)$ are six spin-1 generators of $SU(2)_k \times SU(2)_N$ and $T(z)$ is the spin-2 stress energy tensor. We will use these explicit results written in terms of the Wolf space coset fields in this paper all the times.

Finally the three almost complex structures (satisfying (2.5)) using $4N \times 4N$ matrices are given by

$$
h^{1}_{\bar{a}\bar{b}} = \left(\begin{array}{cccc} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{array}\right), \quad h^{2}_{\bar{a}\bar{b}} = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right), \qquad h^{3}_{\bar{a}\bar{b}} \equiv h^{1}_{\bar{a}\bar{c}}\, h^{2\bar{c}}{}_{\bar{b}}, \qquad (2.7)
$$

where each entry in (2.7) is $N \times N$ matrix. One sees that these complex structures occur in the above 11 currents. The defining OPE equations for the 11 currents are given in Appendix B for convenience. Note that the corresponding relations in different embedding appear in Appendix B of [2].

2.3 The higher spin currents in the Wolf space coset (2.2)

The 16 lowest higher spin currents have the following four $\mathcal{N}=2$ multiplets with spin contents

$$
\begin{aligned}
\left(1,\frac{3}{2},\frac{3}{2},2\right) &: \quad (T^{(1)}, T_{+}^{(\frac{3}{2})}, T_{-}^{(\frac{3}{2})}, T^{(2)}), \\
\left(\frac{3}{2},2,2,\frac{5}{2}\right) &: \quad (U^{(\frac{3}{2})}, U_{+}^{(2)}, U_{-}^{(2)}, U^{(\frac{5}{2})}), \\
\left(\frac{3}{2},2,2,\frac{5}{2}\right) &: \quad (V^{(\frac{3}{2})}, V_{+}^{(2)}, V_{-}^{(2)}, V^{(\frac{5}{2})}), \\
\left(2,\frac{5}{2},\frac{5}{2},3\right) &: \quad (W^{(2)}, W_{+}^{(\frac{5}{2})}, W_{-}^{(\frac{5}{2})}, W^{(3)}). \qquad (2.8)
\end{aligned}
$$

single index notation

$$
\begin{aligned}
G_{11}(z) &= \frac{1}{\sqrt{2}}(G^1 - iG^2)(z), \qquad G_{12}(z) = -\frac{1}{\sqrt{2}}(G^3 - iG^0)(z), \\
G_{22}(z) &= \frac{1}{\sqrt{2}}(G^1 + iG^2)(z), \qquad G_{21}(z) = -\frac{1}{\sqrt{2}}(G^3 + iG^0)(z).
\end{aligned}
$$

The corresponding 16 higher spin currents in different basis will appear in section 4. The higher spin-1 current which will be important in the linear version also is [7]

$$T^{(1)}(z) \;=\; -\frac{1}{2(k+N+2)}\, d^0_{\bar{a}\bar{b}}\, f^{\bar{a}\bar{b}}{}_c V^c(z) + \frac{k}{2(k+N+2)^2}\, d^0_{\bar{a}\bar{b}}\, Q^{\bar{a}}\, Q^{\bar{b}}(z), \tag{2.9}$$

where the rank-two tensor $d^0_{\bar{a}\bar{b}}$ is antisymmetric and satisfies following properties

$$d^0_{\bar{a}\bar{c}}\, d^{0\bar{c}}{}_{\bar{b}} = g_{\bar{a}\bar{b}}, \qquad d^0_{\bar{a}\bar{b}}\, f^{\bar{b}}{}_{\bar{c}d} = d^0_{\bar{c}\bar{b}}\, f^{\bar{b}}{}_{\bar{a}d}. \tag{2.10}$$

The tensorial structure in (2.9) is the same as the one in [2]. Furthermore, the $4N \times 4N$ matrix representation of $d^0_{\bar{a}\bar{b}}$ satisfying (2.10) is

$$d^0_{\bar{a}\bar{b}} = \left(\begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right). \tag{2.11}$$

Each entry is a $N \times N$ matrix as before. Again the corresponding d tensor in [2] was appeared in Appendix B of [2].

Let us define the four higher spin-$\frac{3}{2}$ currents $G'^{\mu}(z)$ from the first order pole of the following OPE

$$G^{\mu}(z)\, T^{(1)}(w) \;=\; \frac{1}{(z-w)}\, G'^{\mu}(w) + \cdots. \tag{2.12}$$

[7] There is also normalization with overall sign where one has the following OPE

$$T^{(1)}(z)T^{(1)}(w) = \frac{1}{(z-w)^2}\left[\frac{2Nk}{N+k+2}\right] + \cdots.$$

One can introduce the $U(1)$ current $U(z)$ described in [16] from this higher spin-1 current. See also (4.20) and (4.38) of [16]. Then one can easily see that the corresponding their $U(z)$ is given by

$$\begin{aligned} U(z) &= (k+N+2)\left[-\frac{1}{2(k+N+2)}\, d^0_{\bar{a}\bar{b}}\, f^{\bar{a}\bar{b}}{}_c V^c(z)\right] - \frac{(N+2)(k+N+2)}{k}\left[\frac{k}{2(k+N+2)^2}\, d^0_{\bar{a}\bar{b}}\, Q^{\bar{a}}\, Q^{\bar{b}}(z)\right] \\ &= -\frac{1}{2}\, d^0_{\bar{a}\bar{b}}\, f^{\bar{a}\bar{b}}{}_c V^c(z) - \frac{(N+2)}{2(k+N+2)}\, d^0_{\bar{a}\bar{b}}\, Q^{\bar{a}}\, Q^{\bar{b}}(z). \end{aligned}$$

The OPEs between $U(z)$ and the coset components of $\mathcal{N}=1$ Kac-Moody currents $Q^a(w)$ and $V^b(w)$ satisfy the following OPEs

$$\begin{aligned} U(z)\left(\begin{array}{c} Q^{\bar{A}} \\ Q^{\bar{A}^*} \end{array}\right)(w) &= \mp\frac{1}{(z-w)}(N+2)\left(\begin{array}{c} Q^{\bar{A}} \\ Q^{\bar{A}^*} \end{array}\right)(w) + \cdots, \\ U(z)\left(\begin{array}{c} V^{\bar{A}} \\ V^{\bar{A}^*} \end{array}\right)(w) &= \mp\frac{1}{(z-w)}(N+2)\left(\begin{array}{c} V^{\bar{A}} \\ V^{\bar{A}^*} \end{array}\right)(w) + \cdots. \end{aligned}$$

Note that the OPEs between $U(z)$ and the 11 currents in (2.6) are regular. One can read off the corresponding $U(1)$ charges in the right hand side. Then the $T^{(1)}$ charges for the coset fields are given by $\pm\frac{k}{(k+N+2)}$ and $\mp\frac{(N+2)}{(k+N+2)}$ respectively.

Then the first order pole in (2.12) provides

$$G'^{\mu}(z) \;=\; \frac{i}{(k+N+2)}\, d^{\mu}_{\bar{a}\bar{b}}\, Q^{\bar{a}}\, V^{\bar{b}}(z), \tag{2.13}$$

where $d^{\mu}_{\bar{a}\bar{b}} \equiv d^{0\bar{c}}_{\bar{a}}\, h^{\mu}_{\bar{c}\bar{b}}$ and $h^{0}_{\bar{a}\bar{b}} \equiv g_{\bar{a}\bar{b}}$ with (2.11). These four independent higher spin-$\frac{3}{2}$ currents also appear in the linear version in section 4.

Then the remaining 15 currents of (2.8) can be written in terms of $\mathcal{N}=1$ Kac-Moody currents $V^a(z), Q^{\bar{b}}(z)$, the three almost complex structures $h^i_{\bar{a}\bar{b}}$, antisymmetric rank-two tensor $d^0_{\bar{a}\bar{b}}$ and symmetric rank-two tensors $d^i_{\bar{a}\bar{b}}(\equiv d^{0\bar{c}}_{\bar{a}}\, h^i_{\bar{c}\bar{b}})$ as in [2].

3 Three-point functions in an extension of large $\mathcal{N}=4$ nonlinear superconformal algebra

This section describes the three-point functions with scalars for the currents of spins $s=1,2$ and the higher spin currents of spins $s=1,2,3$ explained in previous section. The large N limit is defined by [1]

$$N, k \rightarrow \infty, \quad \lambda \equiv \frac{N+1}{N+k+2} \quad \text{fixed}. \tag{3.1}$$

As described in the introduction, there are two simplest states $|(f;0)>$ and $|(0;f)>$ in [1]. These two representations (and their conjugate representations) play an important role in $\mathcal{N}=4$ holographic duality which is conjectured in the large N 't Hooft limit (3.1).

3.1 Eigenvalue equations for spin-2 current in an $\mathcal{N}=4$ nonlinear superconformal algebra

Let us focus on the eigenvalue equations for the stress energy tensor (2.6) acting on the above two states.

3.1.1 Eigenvalue equation for spin-2 current acting on the state $|(f;0)>$

As described in the introduction, the terms containing the fermionic spin-$\frac{1}{2}$ currents $Q^a(z)$ do not contribute to the eigenvalue equation when we calculate the zero mode eigenvalues for the bosonic spin-s current $J^{(s)}(z)$ acting on the state $|(f;0)>$. The zero mode of the spin-1 current satisfies the commutation relation of the underlying finite dimensional Lie algebra $SU(N+2)$. For the state $|(f;0)>$, the generator T_{a^*} corresponds to the zero mode V^a_0 as follows (See also [18]):

$$V^a_0|(f;0)>= T_{a^*}|(f;0)> . \tag{3.2}$$

Then the eigenvalues are encoded in the last 2×2 diagonal matrix. Note that the nonvanishing components of the metric are given by $g_{aa^*}=1$ in Appendix (A.7).

For example, we can calculate the conformal dimension of $|(f;0)>$ when $N=3$. The explicit form for the stress energy tensor is given by (2.6). The only $Q^a(z)$ independent terms are given by the first term and the $A^{+i}A^{+i}(z)$ dependent term because the other terms contain $Q^a(z)$ explicitly. Then the eigenvalue equation for the zero mode of the spin-2 current acting on the state $|(f;0)>$ leads to

$$
\begin{aligned}
T_0|(f;0)> & \sim \left[\frac{1}{2(k+5)}\,V_{\bar{a}}\,V^{\bar{a}}-\frac{1}{(k+5)}\sum_{i=1}^{3}A^{+i}A^{+i}\right]_0 |(f;0)> \\
&= \left[\frac{1}{2(k+5)}\left(\sum_{a=1}^{6}V^aV^{a^*}+\sum_{a=1}^{6}V^{a^*}V^a\right)\right]_0 |(f;0)> +\frac{1}{(k+5)}\left[-\sum_{i=1}^{3}A^{+i}A^{+i}\right]_0 |(f;0)> \\
&= \left[\frac{1}{2(k+5)}\left(\sum_{a=1}^{6}T_{a^*}T_a+\sum_{a=1}^{6}T_aT_{a^*}\right)\right]|(f;0)> +\frac{1}{(k+5)}l^+(l^++1)|(f;0)> \\
&= \frac{1}{2(k+5)}\left(\begin{array}{ccc|cc} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)|(f;0)> +\frac{1}{(k+5)}\frac{3}{4}|(f;0)>=\frac{9}{4(k+5)}|(f;0)>,
\end{aligned}
\tag{3.3}
$$

where $\sim$ in the first line of (3.3) means that we ignore the terms including $Q^a(z)$. In the second line, the summation over the coset indices $\bar{a}=1,2,\cdots,6,1^*,2^*,\cdots,6^*$ is taken explicitly. In the third line, the corresponding $SU(5)$ generators using the condition (3.2) are replaced and moreover the eigenvalue equation for the zero mode of the quadratic spin-1 currents is used where l^+ is the spin of the affine $SU(2)$ algebra. In the fourth line, the $SU(5)$ matrix product is done where the first six terms contribute to the first diagonal elements, 2 and the last six terms contribute to the last diagonal elements, 3. Because the lower 2×2 diagonal matrix corresponding to the zero mode of quadratic spin-1 currents has two eigenvalues $\frac{3}{4}$, the spin l^+ can be identified with $\frac{1}{2}$ [8].

Now we want to obtain the general N dependence for the above eigenvalue equation. From the similar calculations for $N=5,7,9$ where all the (higher spin) currents are known explicitly, we can find N-dependence of the corresponding $(N+2)\times(N+2)$ matrix. In the generalization of third line of (3.3), the first $2N$ terms contribute to the first N diagonal

[8] The highest weight states of the large $\mathcal{N}=4$ (non)linear superconformal algebra can be characterized by the conformal dimension h and two (iso)spins $l^{\pm}$ of $SU(2)\times SU(2)$ [6]

$$
\left[-\sum_{i=1}^{3}A^{+i}A^{+i}\right]_0 |\mathrm{hws}>=l^+(l^++1)\,|\mathrm{hws}>, \qquad \left[-\sum_{i=1}^{3}A^{-i}A^{-i}\right]_0 |\mathrm{hws}>=l^-(l^-+1)\,|\mathrm{hws}>. \tag{3.4}
$$

elements, 2 and the last $2N$ terms contribute to the last two diagonal elements, N. That is, first N diagonal elements are given by 2 and the remaining last two diagonal elements are given by N. The eigenvalues $\frac{3}{4}$ obtained in the footnote 8 hold for any generic N. The total contribution is given by $(\frac{N}{2}+\frac{3}{4})$ multiplied by an obvious overall factor $\frac{1}{(k+N+2)}$. Therefore, the eigenvalue equation (for generic N) for the zero mode of spin-2 current which provides the conformal dimension for the state $|(f;0)>$ is given by

$$T_0|(f;0)> \;\; = \;\; \left[\frac{(2N+3)}{4(k+N+2)}\right]|(f;0)>, \tag{3.5}$$

where the eigenvalue is the same value as $h(f;0)$ given in [1]. One can also check that this leads to the following reduced eigenvalue equation $T_0|(f;0)>=\frac{\lambda}{2}|(f;0)>$ under the large N 't Hooft limit (3.1).

3.1.2 Eigenvalue equation for spin-2 current acting on the state $|(0;f)>$

Let us move on the next simplest representation. When we calculate the eigenvalue equations for the state $|(0;f)>$, because the state is given by [1, 6]

$$|(0;f)>=\frac{1}{\sqrt{k+N+2}}\,Q^{\bar{A}^*}_{-\frac{1}{2}}|0>, \qquad \bar{A}^*=1^*,2^*,\cdots,2N^*, \tag{3.6}$$

the OPEs between the (higher spin) currents $J^{(s)}(z)$ and $Q^{\bar{A}^*}(w)$ are needed. We need only the coefficient of highest order pole $\frac{1}{(z-w)^s}$ in the OPEs. The lower singular terms do not contribute to the zero mode eigenvalue equations. Let us denote the highest order pole as follows [20, 21]:

$$J^{(s)}(z)\,Q^{\bar{A}^*}(w)|_{\frac{1}{(z-w)^s}}=j(s)\,Q^{\bar{A}^*}(w), \tag{3.7}$$

For example, in $G=SU(5)$, the expressions (2.6) imply that

$$\left[-\sum_{i=1}^{3}A^{+i}A^{+i}\right]_0|(f;\star)>=\left(\begin{array}{ccc|cc} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&\frac{3}{4}&0\\ 0&0&0&0&\frac{3}{4}\end{array}\right)|(f;\star)>, \quad \left[-\sum_{i=1}^{3}A^{-i}A^{-i}\right](z)\,Q^{\bar{A}^*}(w)|_{\frac{1}{(z-w)^2}}=\frac{3}{4}Q^{\bar{A}^*}(w),$$

where the representation $\star=0$ (trivial representation) or f (fundamental representation) of $SU(5)$. Note that although all of these expressions are written in terms of the fields without boldface (in the nonlinear version), it is also true that all of these can be replaced by the fields with boldface (in the linear version). We can see $l^+(f;0)=\frac{1}{2}$ (from the two eigenvalues $\frac{3}{4}$), $l^+(f;f)=0$ (from the three eigenvalues 0) and $l^-(0;f)=\frac{1}{2}$ (from the coefficient of the second order pole $\frac{3}{4}$). Then the state $|(f;0)>$ has $l^+=\frac{1}{2}$, $l^-=0$, the state $|(0;f)>$ has $l^+=0$, $l^-=\frac{1}{2}$ and the state $|(f;f)>$ has $l^{\pm}=0$. The eigenvalues for l^- will be explained in next subsection. Note the (-1) sign in the left hand side of (3.4) comes from the anti-hermitian property [6, 19].

where $j(s)$ stands for the corresponding coefficient of highest order pole. One can write down $J_0^{(s)}|(0;f)>$ as $\frac{1}{\sqrt{k+N+2}}[J_0^{(s)}, Q_{-\frac{1}{2}}^{\bar{A}*}]|0>$ and this commutator acting on the vacuum can be written in terms of $\frac{1}{\sqrt{k+N+2}} j(s)\, Q_{-\frac{1}{2}}^{\bar{A}*}|0>$. Then one obtains the following eigenvalue equation for the zero mode of the spin-s current together with (3.6) and (3.7)

$$J_0^{(s)}|(0;f)>= j(s)|(0;f)>, \tag{3.8}$$

where the explicit relation between the current and its mode is given by $J^{(s)}(z) = \sum_{n=-\infty}^{\infty} \frac{J_n^{(s)}}{z^{n+s}}$. Therefore, in order to determine the above eigenvalue $j(s)$, one should calculate the explicit OPEs between the corresponding (higher spin) currents and the spin-$\frac{1}{2}$ current and read off the highest order pole.

Let us consider the eigenvalue equation for the spin-2 current acting on the above state. Since the OPE between the spin-1 current $A^{+i}(z)$ and the spin-$\frac{1}{2}$ current $Q^{\bar{A}*}(w)$ is regular, the terms containing $A^{+i}(z)$ in $T(z)$ of (2.6) do not contribute to the highest order pole. Furthermore the terms containing the spin-1 current $V^a(z)$ do not contribute to the highest order pole. Therefore, the relevant terms in $T(z)$ are given by purely the spin-$\frac{1}{2}$ current dependent terms (by ignoring the spin-1 current dependent terms completely). Then the conformal dimension of $|(0;f)>$ is

$$\begin{aligned} T_0|(0;f)> \quad &\sim \quad \left[\frac{k}{2(k+N+2)^2} Q_{\bar{a}} \partial Q^{\bar{a}} - \frac{1}{(k+N+2)} \sum_{i=1}^{3} A^{-i} A^{-i}\right]_0 |(0;f)> \\ &= \left[\frac{k}{2(k+N+2)^2} Q_{\bar{a}} \partial Q^{\bar{a}}\right]_0 |(0;f)> + \frac{1}{(k+N+2)} \left[-\sum_{i=1}^{3} A^{-i} A^{-i}\right]_0 |(0;f)> \\ &= \frac{k}{2(k+N+2)} |(0;f)> + \frac{1}{(k+N+2)} l^-(l^-+1)|(0;f)> \\ &= \left[\frac{(2k+3)}{4(N+k+2)}\right] |(0;f)> . \end{aligned} \tag{3.9}$$

In the first line of (3.9), the spin-1 current dependent terms are ignored. In the second line, the zero modes for each term are taken. In the third line, we have used the fact that the eigenvalue equation $[Q_{\bar{a}} \partial Q^{\bar{a}}]_0 |(0;f)>= (k+N+2)|(0;f)>$ (see (3.8)) can be obtained because the highest order pole gives the corresponding eigenvalue $Q_{\bar{a}} \partial Q^{\bar{a}}(z)\, Q^{\bar{A}*}(w)|_{\frac{1}{(z-w)^2}} = (k+N+2) Q^{\bar{A}*}(w)$ (see (3.7)) which can be checked from the defining relation in (2.1). Furthermore, the characteristic eigenvalue equation for the affine $SU(2)$ algebra described in the footnote 8 is used. The total contribution is given by $(\frac{k}{2}+\frac{3}{4})$ multiplied by the overall factor $\frac{1}{(N+k+2)}$. The above eigenvalue is exactly the same as the $h(0;f)$ described in [1]. Under the large N 't Hooft limit (3.1), the eigenvalue equation implies that $T_0|(0;f)>= \frac{1}{2}(1-\lambda)|(0;f)>$.

There exists $N \leftrightarrow k$ and $0 \leftrightarrow f$ symmetry between the above two eigenvalue equations (3.5) and (3.9). In the classical symmetry (in the large N 't Hooft limit), this is equivalent to the exchange of $\lambda \leftrightarrow (1-\lambda)$ and $0 \leftrightarrow f$. The U-charge corresponding to twice of $\hat{u}$ charge in [1] can be determined as follows :

$$U_0|(f;0)>= -N|(f;0)>, \qquad U_0|(0;f)>= (N+2)|(0;f)>, \tag{3.10}$$

where we use the same normalization as in [16]. The second relation can be seen from the explicit OPE result in footnote 7 where the eigenvalue $(N+2)$ appears in the OPE between $U(z)$ and $Q^{\bar{A}*}(w)$. The detailed description for these eigenvalue equations will be given in next subsection [9].

3.2 Eigenvalue equation for higher spin currents of spins $1, 2$ and 3

Now one can consider the eigenvalue equations for the higher spin currents by following the descriptions in previous subsection.

3.2.1 Eigenvalue equation for higher spin-1 current acting on the states $|(f;0)>$ and $|(0;f)>$

From the explicit expression in (2.9), one starts with the first term which does not contain the spin-$\frac{1}{2}$ current, applies to the condition (3.2) and read off the last two diagonal matrix elements for $N=3$. It turns out that

$$T_0^{(1)}|(f;0)> \;\; \sim \;\; \left[-\frac{1}{2(k+5)}\, d^0_{\bar{a}\bar{b}}\, f^{\bar{a}\bar{b}}{}_c V^c\right]_0 |(f;0)>$$

[9] Explicitly for $N=3$, one has $U(z) = \frac{1}{4}(3\sqrt{10}+5i\sqrt{6})V^{12}(z) + \frac{1}{4}(3\sqrt{10}-5i\sqrt{6})V^{12^*}(z) + \frac{5}{(5+k)}\sum_{a=1}^{6} Q^a Q^{a^*}(z)$. As done in (3.3), one can construct the following eigenvalue equation for $G = SU(5)$,

$$U_0|(f;\star)>= \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right)|(f;\star)>,$$

where the representation $\star$ is trivial representation 0 or fundamental representation f in $SU(5)$. The eigenvalues -3 living in the lower 2×2 diagonal matrix can be generalized to $-N$ according to the next subsection. By reading off the the first three eigenvalues 2 which is valid for generic N, the U-charge for the state $|(f;f)>$ is given by 2 [1]. Furthermore one can determine the conformal dimension for the 'light' state $|(f;f)>$ by using (3.3) and its generalization. Since $l^+(f;f)=0$ (from the analysis in the footnote 8), the eigenvalue equation is $T_0|(f;f)>= \frac{1}{2(k+N+2)}\times 2|(f;f)>= \frac{1}{(k+N+2)}|(f;f)>$ observed in [1]. Under the large N 't Hooft limit, one has $T_0|(f;f)>= \frac{\lambda}{(N+1)}|(f;f)>$ and this vanishes.

$$
\begin{aligned}
&= \left[-\frac{1}{2(k+5)}\left(-\frac{3\sqrt{10}+5i\sqrt{6}}{2}V^{12}-\frac{3\sqrt{10}-5i\sqrt{6}}{2}V^{12^*}\right)\right]_0 |(f;0)> \\
&= \left[-\frac{1}{2(k+5)}\left(-\frac{3\sqrt{10}+5i\sqrt{6}}{2}T_{12^*}-\frac{3\sqrt{10}-5i\sqrt{6}}{2}T_{12}\right)\right] |(f;0)> \qquad (3.11) \\
&= -\frac{1}{2(k+5)}\left(\begin{array}{ccc|cc} -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right)|(f;0)>= -\frac{3}{(k+5)}|(f;0)> .
\end{aligned}
$$

In the first line of (3.11), one ignores the spin-$\frac{1}{2}$ dependent part. In the second line, one can substitute the d tensor and f structure constant for $N=3$ from the section 2. From the third line to the fourth line, the explicit generators are substituted from Appendix A. Miraculously, the final 5×5 matrix is simple diagonal matrix. Finally, the two eigenvalues 6 from the last 2×2 diagonal matrix are taken [10]. How does one obtain the general N behavior for the above eigenvalue equation? One can find N-dependence of above 5×5 matrix by similar calculation for next several values for N:$N=5,7,9$. The $(N+2)\times(N+2)$ matrix in $G=SU(N+2)$ is given by $\mathrm{diag}(-4,\cdots,-4,2N,2N)$ with the obvious overall factor $-\frac{1}{2(N+k+2)}$. The N dependence appears in the last 2×2 diagonal matrix. Therefore the eigenvalue equation for generic N can be summarized by [11]

$$
T_0^{(1)}|(f;0)> \;=\; -\left[\frac{N}{(k+N+2)}\right]|(f;0)> . \qquad (3.12)
$$

The zero mode eigenvalue equation for the state $|(0;f)>$ can be obtained from the explicit OPE between the second term in (2.9) and the spin-$\frac{1}{2}$ current $Q^{\bar{A}^*}(w)$ and one obtains

$$
T_0^{(1)}|(0;f)> \;=\; -\left[\frac{k}{(k+N+2)}\right]|(0;f)> . \qquad (3.13)
$$

It is obvious that there exists a $N\leftrightarrow k$ and $0\leftrightarrow f$ symmetry between the two eigenvalue equations in (3.12) and (3.13) [12]. Under the large N 't Hooft limit (3.1), one obtains the

[10] The matrix T_{12} is given by the nonzero $11,22,33$ diagonal elements $\frac{1}{60}\left(3\sqrt{10}+5i\sqrt{6}\right)$, 44 diagonal element $\frac{1}{20}\left(\sqrt{10}-5i\sqrt{6}\right)$, and 55 diagonal element $-\frac{\sqrt{10}}{5}$ in Appendix A.

[11] One can read off the eigenvalue equation for the 'light' state $|(f;f)>$ by taking the eigenvalues -4 living in the first $N\times N$ diagonal matrix as follows:

$$
T_0^{(1)}|(f;f)> \;=\; -\frac{1}{2(k+N+2)}\times(-4)|(f;f)>=\left[\frac{2}{(k+N+2)}\right]|(f;f)> .
$$

[12] One can find the above two eigenvalue equations by using U-charge introduced in (3.10) indirectly.

following eigenvalue equations

$$\begin{aligned} T_0^{(1)}|(f;0)> &= -\lambda|(f;0)>, \\ T_0^{(1)}|(0;f)> &= -(1-\lambda)|(0;f)>. \end{aligned} \tag{3.14}$$

Compared to the previous U charge in (3.10), the above higher spin-1 current preserves the $N \leftrightarrow k$ symmetry. If one replaces the fundamental representation f with the antifundamental representation $\bar{f}$ in (3.12) and (3.13), then the extra minus signs appear in the right hand side respectively. Note that the corresponding equations are given by $V_0^a|(\bar{f};0)>= -(T_{a^*})^T|(\bar{f};0)>$ and $|(0;\bar{f})>= \frac{1}{\sqrt{k+N+2}} Q^{\bar{A}}_{-\frac{1}{2}}|0>$ associated with (3.2) and (3.6).

3.2.2 Eigenvalue equation for higher spin-2 currents acting on the states $|(f;0)>$ and $|(0;f)>$

In order to understand the eigenvalue equations for the higher spin-2 currents, one should classify the $|(f;0)>$ state into the following two types of column vectors

$$|(f;0)>_+ = (0,\cdots,0,1,0)^T, \qquad |(f;0)>_- = (0,\cdots,0,0,1)^T. \tag{3.15}$$

They are $\mathbf{2}$ under the $SU(2)$ and transform as singlets under the $SU(N)$ characterized by the first N zeros in (3.15). They have nontrivial $U(1)$ charges (3.10).

On the other hand, the $|(0;f)>$ states are expressed by the following forms

$$\begin{aligned} |(0;f)>_+ &: \frac{1}{\sqrt{k+N+2}} Q^{1^*}_{-\frac{1}{2}}|0>, \qquad \cdots, \qquad \frac{1}{\sqrt{k+N+2}} Q^{N^*}_{-\frac{1}{2}}|0>, \\ |(0;f)>_- &: \frac{1}{\sqrt{k+N+2}} Q^{(N+1)^*}_{-\frac{1}{2}}|0>, \qquad \cdots, \qquad \frac{1}{\sqrt{k+N+2}} Q^{(2N)^*}_{-\frac{1}{2}}|0>. \end{aligned} \tag{3.16}$$

They are fundamental representation $\mathbf{N}$ under the $SU(N)$ respectively and transform as a doublet under the $SU(2)$ [13].

This is because the first term (the second term) of $T^{(1)}(z)$ in (2.9) contributes to the zero mode eigenvalue equation for the state $|(f;0)>$ ($|(0;f)>$) respectively. See the footnote 7. By taking the numerical factors correctly, we find that one of them is given by $T_0^{(1)}|(f;0)>= \frac{1}{(k+N+2)} U_0|(f;0)>= -\frac{N}{(k+N+2)}|(f;0)>$ where the spin-$\frac{1}{2}$ dependent term is ignored and the other is given by $T_0^{(1)}|(0;f)>= -\frac{k}{(k+N+2)(N+2)} U_0|(0;f)>= -\frac{k}{(k+N+2)}|(0;f)>$ where the spin-1 dependent term is ignored.

[13] Let us comment on the nontrivial action of the spin-1 currents into the above two states. The result is as follows:

$$\begin{aligned} A_0^{+\pm}|(f;0)>_{\mp} &= -i|(f;0)>_{\pm}, \\ A_0^{-\pm}|(0;f)>_{\mp} &= i|(0;f)>_{\pm}. \end{aligned}$$

One has nontrivial eigenvalues as one applies to one more zero modes at each expression. The corresponding three point functions can be described.

It turns out that the eigenvalue equations for the higher spin-2 (primary) current $T^{(2)}(z)$ acting on (3.16) and (3.15) are summarized by

$$\begin{aligned}
T_0^{(2)}|(f;0)>_+ &= \left[\frac{N(2Nk+2N-k)}{2(N+k+2)(2Nk+N+k)}\right]|(f;0)>_+, \\
T_0^{(2)}|(f;0)>_- &= -\left[\frac{Nk(2N+3)}{2(N+k+2)(2Nk+N+k)}\right]|(f;0)>_-, \\
T_0^{(2)}|(0;f)>_+ &= \left[\frac{k(2kN+2k-N)}{2(N+k+2)(2Nk+N+k)}\right]|(0;f)>_+, \\
T_0^{(2)}|(0;f)>_- &= -\left[\frac{Nk(2k+3)}{2(N+k+2)(2Nk+N+k)}\right]|(0;f)>_- .
\end{aligned} \tag{3.17}$$

These are the first examples where all the eigenvalues appear differently [14]. There exists $N \leftrightarrow k$ and $0 \leftrightarrow f$ symmetry in (3.17) such that

$$\left[T_0^{(2)}|(f;0)>_\pm\right]_{N\leftrightarrow k, 0\leftrightarrow f} = T_0^{(2)}|(0;f)>_\pm . \tag{3.18}$$

The states $|(f;0)>_\pm$ are changed into $|(0;f)>_\pm$ and vice versa. Furthermore, if one replaces the fundamental representation f with the antifundamental representation $\bar{f}$ in (3.17), then the right hand sides remain unchanged.

The higher spin-2 primary current $\tilde{W}^{(2)}(z)$ was defined in [2] as follows:

$$\tilde{W}^{(2)}(z) \equiv \left[W^{(2)} - \frac{2kN}{(k+N+2kN)}T\right](z). \tag{3.19}$$

Then the corresponding eigenvalue equations from (3.15), (3.16) and (3.19) can be obtained as follows:

$$\begin{aligned}
\tilde{W}_0^{(2)}|(f;0)>_+ &= -\left[\frac{Nk(2N+3)}{2(N+k+2)(2Nk+N+k)}\right]|(f;0)>_+, \\
\tilde{W}_0^{(2)}|(f;0)>_- &= \left[\frac{N(2Nk+2N-k)}{2(N+k+2)(2Nk+N+k)}\right]|(f;0)>_-, \\
\tilde{W}_0^{(2)}|(0;f)>_+ &= \left[\frac{k(2Nk+2k-N)}{2(N+k+2)(2Nk+N+k)}\right]|(0;f)>_+, \\
\tilde{W}_0^{(2)}|(0;f)>_- &= -\left[\frac{Nk(2k+3)}{2(N+k+2)(2Nk+N+k)}\right]|(0;f)>_- .
\end{aligned} \tag{3.20}$$

There exists a $N \leftrightarrow k$ symmetry in (3.20) such that

$$\left[\tilde{W}_0^{(2)}|(f;0)>_\pm\right]_{N\leftrightarrow k, 0\leftrightarrow f, +\leftrightarrow -} = \tilde{W}_0^{(2)}|(0;f)>_\mp . \tag{3.21}$$

[14] More precisely, the previous eigenvalue equations (3.5), (3.9), (3.12) and (3.13) can be rewritten as $T_0|(f;0)>_\pm = \frac{(2N+3)}{4(N+k+2)}|(f;0)>_\pm$, $T_0|(0;f)>_\pm = \frac{(2k+3)}{4(N+k+2)}|(0;f)>_\pm$, $T_0^{(1)}|(f;0)>_\pm = -\frac{N}{(N+k+2)}|(f;0)>_\pm$, and $T_0^{(1)}|(0;f)>_\pm = -\frac{k}{(N+k+2)}|(0;f)>_\pm$ respectively.

Note that the states $|(f;0)>_\pm$ are changed into $|(0;f)>_\mp$ and vice versa. Furthermore, if one replaces the fundamental representation f with the antifundamental representation $\bar{f}$ in (3.20), then the right hand sides remain unchanged.

One can easily see that there exist the precise relations between the above eigenvalue equations as follows:

$$\begin{aligned}
\Big[T_0^{(2)}|(f;0)>_\pm\Big]_{+\leftrightarrow -} &= \tilde{W}_0^{(2)}|(f;0)>_\mp, \\
T_0^{(2)}|(0;f)>_\pm &= \tilde{W}_0^{(2)}|(0;f)>_\pm . \qquad (3.22)
\end{aligned}$$

Furthermore, under the large N 't Hooft limit (3.1), the above eigenvalue equations (3.17) and (3.20) (or (3.22)) become

$$\begin{aligned}
T_0^{(2)}|(f;0)>_\pm &= \pm\frac{1}{2}\lambda|(f;0)>_\pm, \\
T_0^{(2)}|(0;f)>_\pm &= \pm\frac{1}{2}(1-\lambda)|(0;f)>_\pm, \\
\tilde{W}_0^{(2)}|(f;0)>_\pm &= \mp\frac{1}{2}\lambda|(f;0)>_\pm, \\
\tilde{W}_0^{(2)}|(0;f)>_\pm &= \pm\frac{1}{2}(1-\lambda)|(0;f)>_\pm . \qquad (3.23)
\end{aligned}$$

Up to the overall signs, these relations (3.23) behave similarly as the ones in the spin-2 current described in the subsection 3.1.

For the other spin-2 currents one obtains the following nonzero results [15]

$$\begin{aligned}
\Big[U_+^{(2)}\Big]_0|(0;f)>_+ &= \left[\frac{k}{(N+k+2)}\right]|(0;f)>_- \rightarrow (1-\lambda)|(0;f)>_-, \\
\Big[U_-^{(2)}\Big]_0|(f;0)>_+ &= -\left[\frac{N}{(N+k+2)}\right]|(f;0)>_- \rightarrow -\lambda|(f;0)>_-, \\
\Big[V_+^{(2)}\Big]_0|(f;0)>_- &= \left[\frac{N}{(N+k+2)}\right]|(f;0)>_+ \rightarrow \lambda|(f;0)>_+, \\
\Big[V_-^{(2)}\Big]_0|(0;f)>_- &= -\left[\frac{k}{(N+k+2)}\right]|(0;f)>_+ \rightarrow -(1-\lambda)|(0;f)>_+ . \qquad (3.24)
\end{aligned}$$

In (3.24), one sees that as one multiplies the zero modes further, there exist nonzero eigenvalue equations which will be discussed in next subsection (the footnote 19) [16].

[15] More precisely, one has

$$\begin{aligned}
\Big[U_+^{(2)}\Big]_0\frac{1}{\sqrt{N+k+2}}Q^{a^*}_{-\frac{1}{2}}|0> &= \left[\frac{k}{(N+k+2)}\right]\frac{1}{\sqrt{N+k+2}}Q^{(a+N)^*}_{-\frac{1}{2}}|0>, \\
\Big[V_-^{(2)}\Big]_0\frac{1}{\sqrt{N+k+2}}Q^{(a+N)^*}_{-\frac{1}{2}}|0> &= -\left[\frac{k}{(N+k+2)}\right]\frac{1}{\sqrt{N+k+2}}Q^{a^*}_{-\frac{1}{2}}|0>,
\end{aligned}$$

where $a = 1,2,\cdots,N$.

[16] One can easily check the eigenvalue equations for the higher spin-2 current $W^{(2)}$ which is a quasiprimary

3.2.3 Eigenvalue equation for the higher spin-3 current acting on the states $|(f;0)>$ and $|(0;f)>$

It turns out that the eigenvalue equations of the zero mode of the higher spin-3 current $W^{(3)}(z)$ are described as [17]

$$
\begin{aligned}
W_0^{(3)}|(f;0)>_+ &= \left[\frac{N(2N+k+1)(12Nk+10k+4N-1)}{3(N+k+2)^2(6Nk+5N+5k+4)}\right]|(f;0)>_+,\\
W_0^{(3)}|(f;0)>_- &= \left[\frac{N(24N^2k+12Nk^2+8N^2+10k^2+48Nk-6N+43k+1)}{3(N+k+2)^2(6Nk+5N+5k+4)}\right]|(f;0)>_-,\\
W_0^{(3)}|(0;f)>_+ &= -\left[\frac{k(24k^2N+12kN^2+8k^2+10N^2+48kN-6k+43N+1)}{3(N+k+2)^2(6Nk+5N+5k+4)}\right]|(0;f)>_+,\\
W_0^{(3)}|(0;f)>_- &= -\left[\frac{k(2k+N+1)(12Nk+10N+4k-1)}{3(N+k+2)^2(6Nk+5N+5k+4)}\right]|(0;f)>_- . \qquad (3.25)
\end{aligned}
$$

There is a $N \leftrightarrow k$ symmetry in (3.25) such that

$$\left[W_0^{(3)}|(f;0)>_\pm\right]_{N\leftrightarrow k,0\leftrightarrow f,+\leftrightarrow -} = -W_0^{(3)}|(0;f)>_\mp, \qquad (3.26)$$

which looks similar to (3.21) up to the sign. Furthermore, if one replaces the fundamental representation f with the antifundamental representation $\bar{f}$ in (3.25), then the extra minus sign appears in the right hand side respectively.

Under the large N 't Hooft limit (3.1), one has

$$W_0^{(3)}|(f;0)> \;=\; \frac{2}{3}\lambda(1+\lambda)|(f;0)>,$$

field as follows:

$$
\begin{aligned}
W_0^{(2)}|(f;0)>_+ &= 0, \qquad W_0^{(2)}|(f;0)>_- = \left[\frac{N}{(N+k+2)}\right]|(f;0)>_-,\\
W_0^{(2)}|(0;f)>_+ &= \left[\frac{k}{(N+k+2)}\right]|(0;f)>_+, \qquad W_0^{(2)}|(0;f)>_- = 0.
\end{aligned}
$$

Furthermore, its large N 't Hooft limit (3.1) can be summarized by

$$
\begin{aligned}
W_0^{(2)}|(f;0)>_+ &= 0, \qquad W_0^{(2)}|(f;0)>_- = \lambda|(f;0)>_-,\\
W_0^{(2)}|(0;f)>_+ &= (1-\lambda)|(0;f)>_+, \qquad W_0^{(2)}|(0;f)>_- = 0.
\end{aligned}
$$

[17] For $N=3$, the corresponding three diagonal matrix elements are given by $-\frac{4(k-3)(23k+31)}{3(k+5)^2(23k+19)}$, the 44 element, $\frac{(k+7)(46k+11)}{(k+5)^2(23k+19)}$ and the 55 element $\frac{(46k^2+403k+55)}{(k+5)^2(23k+19)}$. The eigenvalue equation for the zero mode of higher spin-3 current acting on the 'light' state can be summarized as follows and its large N 't Hooft limit is also given

$$W_0^{(3)}|(f;f)> \;=\; -\left[\frac{4(k-N)(5N+16+(6N+5)k)}{3(N+k+2)^2(5N+4+(6N+5)k)}\right]|(f;f)> \to \frac{4\lambda(2\lambda-1)}{3N}|(f;f)> \to 0.$$

$$W_0^{(3)}|(0;f)> \;=\; -\frac{2}{3}(1-\lambda)(2-\lambda)|(0;f)>. \tag{3.27}$$

The following relation holds $\left[W_0^{(3)}|(f;0)>\right]_{\lambda\to(1-\lambda),0\leftrightarrow f} = -W_0^{(3)}|(0;f)>$. This is the expected symmetry because of the relation (3.26). The $N \leftrightarrow k$ symmetry corresponds to $\lambda \leftrightarrow (1-\lambda)$ symmetry in the large N limit.

Let us describe the three point functions [18]. From the diagonal modular invariant with pairing up identical representations on the left (holomorphic) and the right (antiholomorphic) sectors [22], one of the primaries is given by $(f;0)\otimes(f;0)$ which is denoted by $\mathcal{O}_+$ and the other is given by $(0;f)\otimes(0;f)$ which is denoted by $\mathcal{O}_-$. Then the three point functions with these two scalars are obtained and their ratios can be written as

$$\begin{aligned}
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+T^{(1)}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-T^{(1)}>} &= \left[\frac{\lambda}{1-\lambda}\right], \\
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+T^{(2)}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-T^{(2)}>} &= \left[\frac{\lambda}{1-\lambda}\right], \\
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+\tilde{W}^{(2)}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-\tilde{W}^{(2)}>} &= -\left[\frac{\lambda}{1-\lambda}\right], \\
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+W^{(3)}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-W^{(3)}>} &= -\left[\frac{\lambda(1+\lambda)}{(1-\lambda)(2-\lambda)}\right].
\end{aligned} \tag{3.28}$$

Compared to the bosonic higher spin AdS/CFT duality in the context of W_N minimal model, the behavior of (3.28) looks similar in the sense that the factor $\frac{\lambda}{(1-\lambda)}$ which is present in the ratios of three point function of higher spin-2 currents appears in the right hand side of the ratio of the three point functions of the higher spin-3 current. Furthermore, the factor $\frac{(1+\lambda)}{(2-\lambda)}$ contribute to the final ratio of the three point functions of higher spin-3 current. Then one expects that the ratio of the three point functions for the higher spin-4 current can be described as $\left[\frac{\lambda(1+\lambda)(2+\lambda)}{(1-\lambda)(2-\lambda)(3-\lambda)}\right]$. Even for the higher spin-s current one can expect that the ratio of the three point functions for the higher spin current of spin-s can be written as $\prod_{n=1}^{s-1}\frac{(n-1+\lambda)}{(n-\lambda)}$ up to sign. Note that the bosonic case has same formula except the numerator of this expression contains n rather than $(n-1)$ [23]. It would be interesting to study this general spin behavior in details [19].

[18] We assume the following normalizations

$$_{\pm}<(\bar{f};0)|(f;0)>_{\pm}={}_{\pm}<(0;\bar{f})|(0;f)>_{\pm}, \qquad {}_{\pm}<(\bar{f};0)|(f;0)>_{\mp}={}_{\pm}<(0;\bar{f})|(0;f)>_{\mp}=0.$$

[19] One describes the following three point functions from (3.24) by specifying the primaries further with $\pm$ indices showing the $SU(2)$ doublet

$$<\overline{\mathcal{O}}_{-,-}\mathcal{O}_{-,+}U_+^{(2)}> \;=\; (1-\lambda)<\overline{\mathcal{O}}_{-,-}\mathcal{O}_{-,-}>,$$

As in the footnote 8, one can calculate the sum of the square of the triplet (corresponding the higher spin-2 currents) in each $SU(2)$ group. For the similar calculations in the nonlinear version where the equation (4.23) of [24] is used, one obtains

$$\begin{aligned}
\left[\sum_{i=1}^{3} \tilde{V}_1^{(1)+i} \tilde{V}_1^{(1)+i}\right]_0 |(f;0)> &= -\left[\frac{12N(5N+4k+2)}{(N+k+2)^2}\right] |(f;0)>, \\
\left[\sum_{i=1}^{3} \tilde{V}_1^{(1)+i} \tilde{V}_1^{(1)+i}\right]_0 |(0;f)> &= -\left[\frac{24k}{(N+k+2)^2}\right] |(0;f)>, \\
\left[\sum_{i=1}^{3} \tilde{V}_1^{(1)-i} \tilde{V}_1^{(1)-i}\right]_0 |(f;0)> &= -\left[\frac{24N}{(N+k+2)^2}\right] |(f;0)>, \\
\left[\sum_{i=1}^{3} \tilde{V}_1^{(1)-i} \tilde{V}_1^{(1)-i}\right]_0 |(0;f)> &= -\left[\frac{12k(5k+4N+2)}{(N+k+2)^2}\right] |(0;f)>. \qquad (3.29)
\end{aligned}$$

There exist the symmetries in $N \leftrightarrow k$ and $0 \leftrightarrow f$. As the large N 't Hooft limits in (3.29) are taken, they become $-12\lambda(4+\lambda)$, $-\frac{24\lambda(1-\lambda)}{N}$, $-\frac{24\lambda^2}{N}$ and $-12(1-\lambda)(5-\lambda)$ respectively.

3.3 Various eigenvalue equations for other states

Using the following identification of the spin-$\frac{1}{2}$ current in [1],

$$\begin{aligned}
\psi^{1,1}(z) &\equiv \frac{1}{\sqrt{k+N+2}} Q^{1^*}(z), \qquad \cdots, \qquad \psi^{N,1}(z) \equiv \frac{1}{\sqrt{k+N+2}} Q^{N^*}(z), \qquad (3.30) \\
\psi^{1,2}(z) &\equiv \frac{1}{\sqrt{k+N+2}} Q^{(N+1)^*}(z), \qquad \cdots, \qquad \psi^{N,2}(z) \equiv \frac{1}{\sqrt{k+N+2}} Q^{(2N)^*}(z),
\end{aligned}$$

one can construct other representations described in [1]. They are symmetric combination or antisymmetric combination of the state with (3.30) as follows:

$$|(0;[2,0,\cdots,0])> = \sum_{(ij)} \psi^{i,\alpha}_{-\frac{1}{2}} \psi^{j,\beta}_{-\frac{1}{2}} |0>, \quad |(0;[0,1,0,\cdots,0])> = \sum_{[ij]} \psi^{i,\alpha}_{-\frac{1}{2}} \psi^{j,\beta}_{-\frac{1}{2}} |0>. \qquad (3.31)$$

Let us calculate the conformal dimensions for these states (3.31). Then one can find the conformal dimensions of above states by using the following results

$$\begin{aligned}
T(z) \sum_{(ij)} \psi^{i,\alpha} \psi^{j,\beta}(w) \Big|_{\frac{1}{(z-w)^2}} &= \left[\frac{k}{(N+k+2)}\right] \sum_{(ij)} \psi^{i,\alpha} \psi^{j,\beta}(w), \\
T(z) \sum_{[ij]} \psi^{i,\alpha} \psi^{j,\beta}(w) \Big|_{\frac{1}{(z-w)^2}} &= \left[\frac{(k+2)}{(N+k+2)}\right] \sum_{[ij]} \psi^{i,\alpha} \psi^{j,\beta}(w). \qquad (3.32)
\end{aligned}$$

$$\begin{aligned}
< \overline{\mathcal{O}}_{+,-} \mathcal{O}_{+,+} U_{-}^{(2)} > &= -\lambda < \overline{\mathcal{O}}_{+,-} \mathcal{O}_{+,-} >, \\
< \overline{\mathcal{O}}_{+,+} \mathcal{O}_{+,-} V_{+}^{(2)} > &= \lambda < \overline{\mathcal{O}}_{+,+} \mathcal{O}_{+,+} >, \\
< \overline{\mathcal{O}}_{-,+} \mathcal{O}_{-,-} V_{-}^{(2)} > &= -(1-\lambda) < \overline{\mathcal{O}}_{-,+} \mathcal{O}_{-,+} >.
\end{aligned}$$

That is the conformal dimension for the former is given by $h(0;[2,0,\cdots,0]) = \frac{k}{(N+k+2)}$ (3.32) and the one for the latter is given by $h(0;[0,1,0,\cdots,0]) = \frac{(k+2)}{(N+k+2)}$ (3.32) which is the same as $(C.8)$ and $(C.9)$ of [1] respectively.

Similarly, one can find the conformal dimension of p-fold antisymmetric product

$$|(0;[0^{p-1},1,0,\cdots,0]) >= \sum_{[i_1 i_2 \cdots i_p]} \psi^{i_1,\alpha_1}_{-\frac{1}{2}} \psi^{i_2,\alpha_2}_{-\frac{1}{2}} \cdots \psi^{i_p,\alpha_p}_{-\frac{1}{2}} |0> . \tag{3.33}$$

From the explicit result in the second order pole $\{T(z) \sum_{[i_1 i_2 \cdots i_p]} \psi^{i_1,\alpha_1} \psi^{i_2,\alpha_2} \cdots \psi^{i_p,\alpha_p}(w)\}_{-2}$, one has

$$\left[\frac{p(p+2+2k)}{4(k+N+2)}\right] \sum_{[i_1 i_2 \cdots i_p]} \psi^{i_1,\alpha_1} \psi^{i_2,\alpha_2} \cdots \psi^{i_p,\alpha_p}(w), \tag{3.34}$$

and one can read off $h(0;[0^{p-1},1,0,\cdots,0]) = \frac{p(p+2+2k)}{4(k+N+2)}$ with (3.33) and (3.34) which is the same as the last equation of Appendix C in [1]. We have checked up to 6-fold products in $G = SU(11)$ case (that is $N = 9$) [20].

There exists other higher representation. Let us check the conformal dimension of $|(f;\bar{f}) >\equiv \frac{1}{\sqrt{N+k+2}} Q^{\bar{A}}_{-\frac{1}{2}} |(f;0)>$ as in [1]. For example, in the coset of $G = SU(3+2)$, one can calculate the following eigenvalue equation

$$\begin{aligned}
T_0|(f;\bar{f})> &= \frac{1}{\sqrt{5+k}} \left(\left[T_0, Q^{\bar{A}}_{-\frac{1}{2}}\right] + Q^{\bar{A}}_{-\frac{1}{2}} T_0 \right) |(f;0)> \\
&= \left[h(0;\bar{f}) + h(f;0)\right] |(f;\bar{f})> \\
&+ \frac{1}{\sqrt{5+k}} Q^{\bar{A}}_{-\frac{1}{2}} \left(\frac{5i\sqrt{6}+3\sqrt{10}}{24(5+k)} V_0^{12} + \frac{-5i\sqrt{6}+3\sqrt{10}}{24(5+k)} V_0^{12^*} \right) |(f;0)> \\
&= \left[h(0;\bar{f}) + h(f;0) + \left(\begin{array}{ccc|cc}
\frac{1}{3(5+k)} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3(5+k)} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3(5+k)} & 0 & 0 \\
\hline
0 & 0 & 0 & -\frac{1}{2(5+k)} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2(5+k)}
\end{array} \right) \right] |(f;\bar{f})> \\
&= \left[\frac{2k+3}{4(5+k)} + \frac{9}{4(5+k)} - \frac{1}{2(5+k)} \right] |(f;\bar{f})>= \frac{1}{2} |(f;\bar{f})>,
\end{aligned} \tag{3.35}$$

where $h(0;\bar{f}) = h(0;f)$. Note that when the operator $\left[T_0, Q^{\bar{A}}_{-\frac{1}{2}}\right]$ acts on the state $|(f;0)>$, the lower order pole (as well as the highest order pole) can contribute to the eigenvalue equation.

[20] As described in [1], the 'short' representations are given by the following conditions

$$[G_{2a}]_{-\frac{1}{2}} |(f;0)>_+ = [G_{1a}]_{-\frac{1}{2}} |(f;0)>_- = 0,$$
$$[G_{a1}]_{-\frac{1}{2}} |(0;\bar{f})>_+ = [G_{a2}]_{-\frac{1}{2}} |(0;\bar{f})>_- = 0,$$

where $a = 1,2$. These correspond to (2.39) of [1].

In the coset theory of $G = SU(N+2)$, the above matrix in (3.35) can be generalized to

$$\frac{1}{(N+k+2)} \operatorname{diag}(\frac{1}{N}, \cdots, \frac{1}{N}, -\frac{1}{2}, -\frac{1}{2}). \tag{3.36}$$

Then the conformal dimension for generic N can be described as

$$\begin{aligned} T_0|(f;\bar{f})> &= \left[h(0;\bar{f}) + h(f;0) - \frac{1}{2(N+k+2)}\right]|(f;\bar{f})> \\ &= \left[\frac{(2k+3)}{4(N+k+2)} + \frac{(2N+3)}{4(N+k+2)} - \frac{1}{2(N+k+2)}\right]|(f;\bar{f})> = \frac{1}{2}|(f;\bar{f})>, \end{aligned} \tag{3.37}$$

where the result of (3.36) is used and this (3.37) was observed in [1].

Therefore, in this section, the ratios of the three point functions can be summarized by (3.28). They can be obtained from the previous relations (3.14), (3.23) and (3.27).

4 The large $\mathcal{N} = 4$ linear superconformal algebra and its extension in the coset theory

For the 16 currents of large $\mathcal{N} = 4$ linear superconformal algebra, the work of [11] leads to the complete expressions in terms of the coset fields. In this section, some of the recapitulation of [11] in our notations are given and one would like to construct the 16 higher spin currents.

4.1 The 16 currents of $\mathcal{N} = 4$ linear superconformal algebra using the Kac-Moody currents

The 16 currents and the 16 higher spin currents are constructed in the following coset theory

$$\frac{G}{H} = \frac{SU(N+2)}{SU(N)}. \tag{4.1}$$

The following three indices are defined in the corresponding group G, subgroup H and the coset $\frac{G}{H}$ respectively

$$\begin{aligned} G \quad &\text{indices} \quad : \quad a, b, c, \cdots, \\ H \quad &\text{indices} \quad : \quad a', b', c', \cdots, \\ \frac{G}{H} \quad &\text{indices} \quad : \quad \tilde{a}, \tilde{b}, \tilde{c}, \cdots. \end{aligned} \tag{4.2}$$

The number of coset indices is given by the difference between $(N+2)^2 - 1$ and $(N^2 - 1)$ and therefore the dimension of the coset is given by $(4N+4)$. For given $(N+2) \times (N+2)$ matrix

the $(4N+4)$ coset indices can be associated with the following locations with asterisk

$$\left(\begin{array}{ccccc|cc} & & & & & * & * \\ & & & & & * & * \\ & & & & & \vdots & \vdots \\ & & & & & * & * \\ & & & & & * & * \\ \hline * & * & \cdots & * & * & * & * \\ * & * & \cdots & * & * & * & * \end{array}\right)_{(N+2)\times(N+2)} . \qquad (4.3)$$

Compared to the previous case in (2.4), there are extra 2×2 matrix corresponding $SU(2)\times U(1)$. One can further divide the linear coset indices (4.2) as $\tilde{a}=(\bar{a},\hat{a})$ where the index $\hat{a}$ associates with the above 2×2 matrix and runs over 4 values. Of course, the remaining $\bar{a}$ index runs over $4N$ values as in an extension of large $\mathcal{N}=4$ nonlinear superconformal algebra in the section 2.

Let us consider four spin-$\frac{3}{2}$ currents of large $\mathcal{N}=4$ linear superconformal algebra. It is known that the spin-$\frac{3}{2}$ current corresponding to the $\mathcal{N}=1$ supersymmetry generator consists of two parts. One of them contains the spin-1 current as well as the spin-$\frac{1}{2}$ current and the other contains the cubic term in the spin-$\frac{1}{2}$ current. See also [25]. By generalizing the two coefficient tensors to possess the three additional supersymmetry indices, one can write down

$$\mathbf{G}^{\mu}(z)=A(k,N)\left[\,h^{\mu}_{\tilde{a}\tilde{b}}\,Q^{\tilde{a}}\,V^{\tilde{b}}+B(k,N)S^{\mu}_{\tilde{a}\tilde{b}\tilde{c}}Q^{\tilde{a}}Q^{\tilde{b}}Q^{\tilde{c}}\right](z), \qquad (\mu=0,1,2,3), \qquad (4.4)$$

where the relative coefficients $A(k,N)\equiv\frac{i}{(k+N+2)}$ and $B(k,N)\equiv-\frac{1}{6(k+N+2)}$ are taken from the $\mathcal{N}=1$ supersymmetry generator and moreover the two tensors are given by $h^{0}_{\tilde{a}\tilde{b}}\equiv g_{\tilde{a}\tilde{b}}$ and $S^{0}_{\tilde{a}\tilde{b}\tilde{c}}\equiv f_{\tilde{a}\tilde{b}\tilde{c}}$ for $\mu=0$ index. The new objects $h^{i}_{\tilde{a}\tilde{b}}$ and $S^{i}_{\tilde{a}\tilde{b}\tilde{c}}$ for other three indices $i=1,2,3$ are undetermined numerical constants [21].

Then one obtains the explicit OPE between the spin-$\frac{3}{2}$ current (4.4) and itself as follows:

$$\begin{aligned}
\mathbf{G}^{\mu}(z)\,\mathbf{G}^{\nu}(w) &= \frac{1}{(z-w)^3}\,A^2\left[-k(k+N+2)\,h^{\mu}_{\tilde{a}\tilde{b}}\,h^{\nu\tilde{a}\tilde{b}}+6B^2(k+N+2)^3 S^{\mu}_{\tilde{a}\tilde{b}\tilde{c}}S^{\nu\tilde{a}\tilde{b}\tilde{c}}\right] \\
&+ \frac{1}{(z-w)^2}\,A^2\left[(k+N+2)\,h^{\mu}_{\tilde{a}\tilde{b}}\,h^{\nu\tilde{a}}{}_{\tilde{d}}\,f^{\tilde{b}\tilde{d}}{}_{e}\,V^{e}+k\,h^{\mu}_{\tilde{a}\tilde{b}}\,h^{\nu\tilde{b}}_{\tilde{c}}\,Q^{\tilde{a}}\,Q^{\tilde{c}}\right. \\
&- \left.18B^2(k+N+2)^2 S^{\mu}_{\tilde{a}\tilde{b}\tilde{c}}S^{\nu\tilde{a}\tilde{b}}{}_{\tilde{d}}\,Q^{\tilde{c}}Q^{\tilde{d}}\right](w) \\
&+ \frac{1}{(z-w)}\,A^2\left[-(k+N+2)\,h^{\mu}_{\tilde{a}\tilde{b}}\,h^{\nu\tilde{a}}{}_{\tilde{d}}\,V^{\tilde{b}}\,V^{\tilde{d}}+k\,h^{\mu}_{\tilde{a}\tilde{b}}\,h^{\nu\tilde{b}}_{\tilde{c}}\,\partial\,Q^{\tilde{a}}\,Q^{\tilde{c}}\right. \\
&- h^{\mu}_{\tilde{a}\tilde{b}}\,h^{\nu}_{\tilde{c}\tilde{d}}f^{\tilde{b}\tilde{d}}{}_{e}\,Q^{\tilde{a}}\,Q^{\tilde{c}}\,V^{e}-6B(k+N+2)S^{(\mu}_{\tilde{a}\tilde{b}\tilde{c}}h^{\nu)\tilde{a}}{}_{\tilde{d}}\,Q^{\tilde{b}}\,Q^{\tilde{c}}\,V^{\tilde{d}}
\end{aligned}$$

[21] One uses the boldface notation for the 16 currents plus 16 higher spin currents in the linear version. For the previous 11 currents and 16 higher spin currents in the nonlinear version, the boldface notation is not used.

$$- \quad 9B^2(k+N+2) S^{\mu}_{\tilde{a}\tilde{b}\tilde{c}} S^{\nu\tilde{a}}{}_{\tilde{d}\tilde{e}} \, Q^{\tilde{b}} Q^{\tilde{c}} Q^{\tilde{d}} Q^{\tilde{e}}$$
$$- \quad 18B^2(k+N+2)^2 S^{\mu}_{\tilde{a}\tilde{b}\tilde{c}} S^{\nu\tilde{a}\tilde{b}}{}_{\tilde{d}} \, \partial Q^{\tilde{c}} Q^{\tilde{d}} \Big] (w) + \cdots . \qquad (4.5)$$

Compared to the similar calculation for the OPE between the spin-$\frac{3}{2}$ current and itself in the nonlinear version, the result in (4.5) contains the S^{μ} tensor dependent terms.

By using the defining $\mathcal{N}=4$ linear superconformal algebra (C.1) and above OPE (4.5), one identifies the 16 currents in terms of spin-1 current and spin-$\frac{1}{2}$ current. From the particular expression when $\mu=\nu=0$, one can read off the spin-2 current explicitly. That is, the first order pole is given by $\mathbf{G}^0(z)\,\mathbf{G}^0(w)|_{\frac{1}{(z-w)}} = 2\mathbf{T}(w)$, and the corresponding expression is obtained from the OPE (4.5). Then one obtains

$$\mathbf{T}(z) = \frac{1}{2(k+N+2)^2} \Big[(k+N+2)\, V_{\tilde{a}}\, V^{\tilde{a}} + k\, Q_{\tilde{a}}\, \partial\, Q^{\tilde{a}} + f_{\tilde{a}\tilde{b}c'}\, Q^{\tilde{a}}\, Q^{\tilde{b}}\, V^{c'}$$
$$+ \frac{1}{2} f_{\tilde{a}\tilde{b}\tilde{c}} f^{\tilde{a}\tilde{b}}{}_{\tilde{d}} \partial Q^{\tilde{c}} Q^{\tilde{d}} + \frac{1}{4(k+N+2)} f_{\tilde{a}\tilde{b}\tilde{e}} f_{\tilde{c}\tilde{d}}{}^{\tilde{e}} Q^{\tilde{a}} Q^{\tilde{b}} Q^{\tilde{c}} Q^{\tilde{d}} \Big] (z). \qquad (4.6)$$

Compared to the corresponding spin-2 current in (2.6), the dummy variables are summed over the extra 4 indices corresponding to the lower 2×2 matrix in (4.3).

From the $V^{\tilde{b}}\, V^{\tilde{d}}(w)$ term of the following relation $\mathbf{G}^{(\mu}(z)\,\mathbf{G}^{\nu)}(w)|_{\frac{1}{(z-w)}} = 2\delta^{\mu\nu}\mathbf{T}(w)$, one has the identity

$$h^{\mu}_{\tilde{a}\tilde{b}}\, h^{\nu\tilde{a}}{}_{\tilde{d}} + h^{\nu}_{\tilde{a}\tilde{b}}\, h^{\mu\tilde{a}}{}_{\tilde{d}} = 2\,\delta^{\mu\nu}\, g_{\tilde{b}\tilde{d}}, \qquad (\mu,\nu=0,1,2,3), \qquad (4.7)$$

which corresponds to (2.19) of [11]. The left hand side of (4.7) comes from the above OPE (4.5) and the right hand side comes from the (4.6). Thus $h^{i}_{\tilde{a}\tilde{b}}$ are almost complex structures.

From the $Q^{\tilde{a}}\, Q^{\tilde{b}}\, V^{\tilde{e}}$ term of $\mathbf{G}^{(\mu}(z)\,\mathbf{G}^{\nu)}(w)|_{\frac{1}{(z-w)}} = 2\delta^{\mu\nu}\mathbf{T}(w)$ [22], the following identities are satisfied

$$h^{(\mu}_{\tilde{a}\tilde{c}} h^{\nu)}_{\tilde{b}\tilde{d}} f^{\tilde{c}\tilde{d}}{}_{\tilde{e}} = h^{\tilde{d}}{}_{\tilde{e}}{}^{(\mu} S^{\nu)}_{\tilde{d}\tilde{a}\tilde{b}},$$
$$h^{i}_{\tilde{a}\tilde{c}} f^{\tilde{c}}{}_{\tilde{b}e'} = h^{i}_{\tilde{b}\tilde{c}} f^{\tilde{c}}{}_{\tilde{a}e'}, \qquad (4.8)$$

which correspond to (2.20) and (2.21) of [11] respectively. The second equation of (4.8) can be seen from the equation (3.11) of [26]. By using the three identities given in (4.7) and (4.8), the complete expression for the $S^{i}_{\tilde{a}\tilde{b}\tilde{c}}$ tensor is as follows:

$$S^{i}_{\tilde{a}\tilde{b}\tilde{c}} = h^{i}_{\tilde{a}\tilde{d}} h^{i}_{\tilde{b}\tilde{e}} h^{i}_{\tilde{c}\tilde{f}} f^{\tilde{d}\tilde{e}\tilde{f}}, \qquad (i=1,2,3), \qquad (4.9)$$

corresponding to (2.27) of [11].

[22] In the nonlinear version of [2], there was no second identity of (4.8). Because there exists a quadratic term for the spin-1 current in the first order pole $\frac{1}{(z-w)}$ of Appendix (B.1), the different identity was required.

Thus, the spin-$\frac{3}{2}$ currents are given by, together with the three almost complex structures and the structure constants where the result in (4.9) is substituted,

$$\begin{aligned}
\mathbf{G}^0(z) &= \frac{i}{(k+N+2)}\left[Q_{\tilde{a}}\,V^{\tilde{a}} - \frac{1}{6(k+N+2)} f_{\tilde{a}\tilde{b}\tilde{c}} Q^{\tilde{a}} Q^{\tilde{b}} Q^{\tilde{c}}\right](z),\\
\mathbf{G}^j(z) &= \frac{i}{(k+N+2)}\left[h^j_{\tilde{a}\tilde{b}}\,Q^{\tilde{a}}\,V^{\tilde{b}} - \frac{1}{6(k+N+2)} h^j_{\tilde{a}\tilde{d}} h^j_{\tilde{b}\tilde{e}} h^j_{\tilde{c}\tilde{f}} f^{\tilde{d}\tilde{e}\tilde{f}} Q^{\tilde{a}} Q^{\tilde{b}} Q^{\tilde{c}}\right](z).
\end{aligned} \quad (4.10)$$

Compared to the corresponding spin-$\frac{3}{2}$ currents in the nonlinear version, the above expressions (4.10) have cubic term in the spin-$\frac{1}{2}$ current and the index $\tilde{a}$ contains 4 indices corresponding to the 2×2 matrix as before.

Let us determine other currents. The six spin-1 currents $\mathbf{A}^{\pm i}(z)$ can be obtained from the second order pole $\frac{1}{(z-w)^2}$ terms of the OPE (4.5) and the corresponding OPE in (C.1) [23],

$$\begin{aligned}
\mathbf{A}^{+i}(z) &= -\frac{1}{4(N+1)}\left[h^i_{\tilde{a}\tilde{b}} f^{\tilde{a}\tilde{b}}{}_c V^c + \frac{1}{(k+N+2)}\left(h^i_{\tilde{c}\tilde{d}} + \frac{1}{2} S^i_{\tilde{a}\tilde{b}\tilde{c}} f^{\tilde{a}\tilde{b}}{}_d\right) Q^{\tilde{c}} Q^{\tilde{d}}\right](z),\\
\mathbf{A}^{-i}(z) &= -\frac{1}{4(k+N+2)} h^i_{\tilde{a}\tilde{b}} Q^{\tilde{a}} Q^{\tilde{b}}(z),
\end{aligned} \quad (4.11)$$

corresponding to (2.36) and (2.37) of [11]. From the second order poles in the OPEs $\mathbf{A}^{\pm i}(z)\,\mathbf{G}^{\mu}(w)$ with (4.10) and (4.11), the four fermionic spin-$\frac{1}{2}$ currents $\mathbf{\Gamma}^{\mu}(z)$ corresponding to (2.43) of [11] can be fixed as follows:

$$\mathbf{\Gamma}^0(z) = -\frac{i}{4(N+1)} h^j_{\tilde{a}\tilde{b}} f^{\tilde{a}\tilde{b}}{}_{\tilde{c}} h^{j\tilde{c}}{}_{\tilde{d}} Q^{\tilde{d}}(z), \qquad \mathbf{\Gamma}^j(z) = -\frac{i}{4(N+1)} h^j_{\tilde{a}\tilde{b}} f^{\tilde{a}\tilde{b}}{}_{\tilde{c}} Q^{\tilde{c}}(z), \quad (4.12)$$

where $j = 1, 2, 3$ and there is no sum over j in the first equation of (4.12). From the OPE $\mathbf{\Gamma}^{\mu}(z)\,\mathbf{G}^{\nu}(w)$ when the index μ is the same as the index ν, the spin-1 current $\mathbf{U}(z)$ corresponding to (2.44) of [11] can be determined by

$$\mathbf{U}(z) = -\frac{1}{4(N+1)} h^j_{\tilde{a}\tilde{b}} f^{\tilde{a}\tilde{b}}{}_{\tilde{c}} h^{j\tilde{c}}{}_{\tilde{d}} \left[V^{\tilde{d}} - \frac{1}{2(k+N+2)} f^{\tilde{d}}{}_{\tilde{e}\tilde{f}} Q^{\tilde{e}} Q^{\tilde{f}}\right](z), \quad (4.13)$$

where there is no sum over an index j. One can easily see that the first (the second) term of (4.13) comes from the OPE between $\mathbf{\Gamma}^{\mu}(z)$ in (4.12) and the spin-1 term (the cubic term in the spin-$\frac{1}{2}$ current) in $\mathbf{G}^{\mu}(w)$ in (4.10) [24].

[23] We have used the following tensor identities,

$$S^{0\tilde{a}\tilde{b}}{}_{\tilde{c}} S^1_{\tilde{a}\tilde{b}\tilde{d}} + S^{2\tilde{a}\tilde{b}}{}_{\tilde{c}} S^3_{\tilde{a}\tilde{b}\tilde{d}} = 4h^1_{\tilde{c}\tilde{d}}, \quad S^{0\tilde{a}\tilde{b}}{}_{\tilde{c}} S^2_{\tilde{a}\tilde{b}\tilde{d}} + S^{3\tilde{a}\tilde{b}}{}_{\tilde{c}} S^1_{\tilde{a}\tilde{b}\tilde{d}} = 4h^2_{\tilde{c}\tilde{d}}, \quad S^{0\tilde{a}\tilde{b}}{}_{\tilde{c}} S^3_{\tilde{a}\tilde{b}\tilde{d}} + S^{1\tilde{a}\tilde{b}}{}_{\tilde{c}} S^2_{\tilde{a}\tilde{b}\tilde{d}} = 4h^3_{\tilde{c}\tilde{d}}.$$

[24] As in the footnote 7, one can calculate the corresponding $\mathbf{U}$ charges for the coset fields as follows:

$$i\mathbf{U}(z)\begin{pmatrix} Q^{\bar{A}} \\ Q^{\bar{A}^*} \end{pmatrix}(w) = \mp\frac{1}{(z-w)}\left[\frac{1}{2}\sqrt{\frac{N+2}{N}}\right]\begin{pmatrix} Q^{\bar{A}} \\ Q^{\bar{A}^*} \end{pmatrix}(w) + \cdots,$$

The defining OPE equations for the 16 currents are given in Appendix C for convenience. Appendix D contains the explicit form for the complex structures.

4.2 The higher spin currents in the coset (4.1)

Let us consider the higher spin currents in an extension of large $\mathcal{N}=4$ linear superconformal algebra. Again, it is crucial to find the lowest higher spin-1 current and it is straightforward to obtain the remaining higher spin currents. Let us denote the 16 higher spin currents as follows:

$$\begin{aligned}
\left(1,\frac{3}{2},\frac{3}{2},2\right) &: (\mathbf{T}^{(1)},\mathbf{T}_+^{(\frac{3}{2})},\mathbf{T}_-^{(\frac{3}{2})},\mathbf{T}^{(2)}),\\
\left(\frac{3}{2},2,2,\frac{5}{2}\right) &: (\mathbf{U}^{(\frac{3}{2})},\mathbf{U}_+^{(2)},\mathbf{U}_-^{(2)},\mathbf{U}^{(\frac{5}{2})}),\\
\left(\frac{3}{2},2,2,\frac{5}{2}\right) &: (\mathbf{V}^{(\frac{3}{2})},\mathbf{V}_+^{(2)},\mathbf{V}_-^{(2)},\mathbf{V}^{(\frac{5}{2})}),\\
\left(2,\frac{5}{2},\frac{5}{2},3\right) &: (\mathbf{W}^{(2)},\mathbf{W}_+^{(\frac{5}{2})},\mathbf{W}_-^{(\frac{5}{2})},\mathbf{W}^{(3)}).
\end{aligned} \tag{4.14}$$

Here the 16 higher spin currents are primary under the stress energy tensor given in (4.6). In the convention of [24], the higher spin-3 current is a quasiprimary current under the stress energy tensor.

4.2.1 The higher spin-1 current

The natural ansatz for the higher spin-1 current is

$$\mathbf{T}^{(1)}(z) = A_a\, V^a(z) + B_{\tilde{a}\tilde{b}}\, Q^{\tilde{a}}\, Q^{\tilde{b}}(z). \tag{4.15}$$

By requiring that the OPEs between the above higher spin-1 current and six spin-1 currents are regular, the higher spin-1 current is a primary under the stress energy tensor $\mathbf{T}(w)$ and the OPEs between the higher spin-1 current and 4 free fermions $\mathbf{\Gamma}^{\mu}(w)$ (and $\mathbf{U}(w)$) are regular [24], all the unknown coefficients in (4.15) are determined completely. Then, it turns out that the higher spin-1 current (4.15) in the linear version is the same as the higher spin-1 current

$$i\mathbf{U}(z)\begin{pmatrix} V^{\bar{A}} \\ V^{\bar{A}^*} \end{pmatrix}(w) = \mp\frac{1}{(z-w)}\left[\frac{1}{2}\sqrt{\frac{N+2}{N}}\right]\begin{pmatrix} V^{\bar{A}} \\ V^{\bar{A}^*} \end{pmatrix}(w)+\cdots.$$

These are proportional to the U charges in the footnote 7.

in the nonlinear version [25]

$$\mathbf{T^{(1)}}(z) = T^{(1)}(z). \tag{4.16}$$

Now the lowest higher spin-1 current is completely fixed and it is straightforward to calculate the remaining higher spin currents as mentioned before.

4.2.2 The higher spin-$\frac{3}{2}$ currents in (4.14)

Let us define the four higher spin-$\frac{3}{2}$ currents $\mathbf{G}'^{\mu}(w)$ from the first order pole of the following OPE

$$\mathbf{G}^{\mu}(z)\,\mathbf{T^{(1)}}(w) \;=\; \frac{1}{(z-w)}\,\mathbf{G}'^{\mu}(w)+\cdots. \tag{4.17}$$

It turns out that the first order pole in (4.17) provides

$$\mathbf{G}'^{\mu}(z) = G'^{\mu}(z), \tag{4.18}$$

which is given by (2.13). This implies that the OPE between the extra terms between $\mathbf{G}^{\mu}(z)$ and $G^{\mu}(z)$ in Appendix (F.1) and higher spin-1 current is regular.

By calculating the OPEs between the spin-$\frac{3}{2}$ currents in (4.10) and the higher spin-1 current in (4.16) and by subtracting the spin-$\frac{3}{2}$ currents in the left hand side with correct coefficients, the following higher spin-$\frac{3}{2}$ currents can be obtained

$$\begin{aligned}
\mathbf{T}_{+}^{(\frac{3}{2})}(z) &= \frac{1}{2}\left(\mathbf{G}'_{21}-\mathbf{G}_{21}\right)(z) = \frac{1}{2}\left(G'_{21}-\mathbf{G}_{21}\right)(z), \\
\mathbf{T}_{-}^{(\frac{3}{2})}(z) &= \frac{1}{2}\left(\mathbf{G}'_{12}+\mathbf{G}_{12}\right)(z) = \frac{1}{2}\left(G'_{12}+\mathbf{G}_{12}\right)(z), \\
\mathbf{U}^{(\frac{3}{2})}(z) &= \frac{1}{2}\left(\mathbf{G}'_{11}-\mathbf{G}_{11}\right)(z) = \frac{1}{2}\left(G'_{11}-\mathbf{G}_{11}\right)(z), \\
\mathbf{V}^{(\frac{3}{2})}(z) &= \frac{1}{2}\left(\mathbf{G}'_{22}+\mathbf{G}_{22}\right)(z) = \frac{1}{2}\left(G'_{22}+\mathbf{G}_{22}\right)(z).
\end{aligned} \tag{4.19}$$

In each last line, the conditions in (4.18) are used. Furthermore, the relations in the footnote 6 between the double index and a single index hold in this case. Compared to the corresponding higher spin-$\frac{3}{2}$ currents in the nonlinear version [2], the four corresponding OPEs generating (4.19) are the same as the ones in the nonlinear version. See also the first four equations in Appendix E. In other words, those four equations remain unchanged if one uses the currents without a boldface.

[25] Of course, we required same normalization with the higher spin-1 current $T^{(1)}$ in the nonlinear version and take the same sign as follows

$$\mathbf{T^{(1)}}(z)\,\mathbf{T^{(1)}}(w) = \frac{1}{(z-w)^2}\left[\frac{2Nk}{N+k+2}\right]+\cdots.$$

4.2.3 The remaining higher spin currents

So far, the higher spin-1 and four higher spin-$\frac{3}{2}$ currents are obtained from (4.16) and (4.19).

How does one obtain the other higher spin currents? Let us consider the six higher spin-2 currents in (4.14). For example, the following higher spin-2 current can be described as follows:

$$\begin{aligned}
\mathbf{U}_{-}^{(2)}(w) &= \{\mathbf{G}_{12}\,\mathbf{U}^{(\frac{3}{2})}\}_{-1}(w) - \frac{1}{2}\partial\{\mathbf{G}_{12}\,\mathbf{U}^{(\frac{3}{2})}\}_{-2}(w) \\
&= \frac{1}{2}\{\mathbf{G}_{12}\,G'_{11}\}_{-1}(w) - \frac{1}{2}\{\mathbf{G}_{12}\,\mathbf{G}_{11}\}_{-1}(w) - \frac{1}{2}\partial\{\mathbf{G}_{12}\,\mathbf{U}^{(\frac{3}{2})}\}_{-2}(w) \\
&= \frac{1}{2}\{\mathbf{G}_{12}\,G'_{11}\}_{-1}(w).
\end{aligned} \tag{4.20}$$

The first line of (4.20) comes from the seventh OPE in Appendix E. In the second line, the equation (4.19) is used. It turns out that the second and third terms are canceled each other. Then the higher spin-2 current can be obtained from the first order pole of the OPE between $\mathbf{G}_{12}(z)$ and $G'_{11}(w)$.

Similarly, the other remaining higher spin-2 currents can be simplified as follows:

$$\begin{aligned}
\mathbf{V}_{+}^{(2)}(w) &= \frac{1}{2}\{\mathbf{G}_{21}\,G'_{22}\}_{-1}(w), \\
\mathbf{V}_{-}^{(2)}(w) &= \frac{1}{2}\{\mathbf{G}_{12}\,G'_{22}\}_{-1}(w), \\
\mathbf{U}_{+}^{(2)}(w) &= \frac{1}{2}\{\mathbf{G}_{21}\,G'_{11}\}_{-1}(w), \\
\mathbf{T}^{(2)}(w) &= \frac{1}{2}\{\mathbf{G}_{21}\,G'_{12}\}_{-1}(w) - \frac{1}{2}\partial\mathbf{T}^{(1)}(w), \\
\mathbf{W}^{(2)}(w) &= \frac{1}{2}\{\mathbf{G}_{11}\,G'_{22}\}_{-1}(w) - \frac{1}{2}\partial\mathbf{T}^{(1)}(w).
\end{aligned} \tag{4.21}$$

In (4.21), because the derivative of higher spin-1 current can be obtained easily, the nontrivial parts for the higher spin currents are given by the OPE $\mathbf{G}_{\alpha\beta}(z)\,G'_{\rho\sigma}(w)$.

Let us write down $\mathbf{G}_{\alpha\beta}(z)$ and $G'_{\rho\sigma}(z)$ explicitly with simplified notations

$$\begin{aligned}
\mathbf{G}_{\alpha\beta}(z) &= \frac{i}{(k+N+2)}\left[h^{\alpha\beta}_{\tilde{a}\tilde{b}}\,Q^{\tilde{a}}\,V^{\tilde{b}} - \frac{1}{6(k+N+2)}S^{\alpha\beta}_{\tilde{a}\tilde{b}\tilde{c}}Q^{\tilde{a}}Q^{\tilde{b}}Q^{\tilde{c}}\right](z), \\
G'_{\rho\sigma}(z) &= \frac{i}{(k+N+2)}d^{\rho\sigma}_{\bar{a}\bar{b}}\,Q^{\bar{a}}\,V^{\bar{b}}(z).
\end{aligned} \tag{4.22}$$

where $\alpha,\beta,\rho,\sigma = 1,2$ and in (4.22) the following notations are introduced

$$h^{11}_{\tilde{a}\tilde{b}} \equiv \frac{1}{\sqrt{2}}\left(h^1 - ih^2\right)_{\tilde{a}\tilde{b}}, \quad h^{12}_{\tilde{a}\tilde{b}} \equiv -\frac{1}{\sqrt{2}}\left(h^3 - ih^0\right)_{\tilde{a}\tilde{b}},$$

$$
\begin{aligned}
h^{22}_{\tilde{a}\tilde{b}} & \equiv \frac{1}{\sqrt{2}}\left(h^1+ih^2\right)_{\tilde{a}\tilde{b}}\,, \quad h^{21}_{\tilde{a}\tilde{b}} \equiv -\frac{1}{\sqrt{2}}\left(h^3+ih^0\right)_{\tilde{a}\tilde{b}}\,, \\
S^{11}_{\tilde{a}\tilde{b}\tilde{c}} & \equiv \frac{1}{\sqrt{2}}\left(S^1-iS^2\right)_{\tilde{a}\tilde{b}\tilde{c}}\,, \qquad S^{12}_{\tilde{a}\tilde{b}\tilde{c}} \equiv -\frac{1}{\sqrt{2}}\left(S^3-iS^0\right)_{\tilde{a}\tilde{b}\tilde{c}}\,, \\
S^{22}_{\tilde{a}\tilde{b}\tilde{c}} & \equiv \frac{1}{\sqrt{2}}\left(S^1+iS^2\right)_{\tilde{a}\tilde{b}\tilde{c}}\,, \qquad S^{21}_{\tilde{a}\tilde{b}\tilde{c}} \equiv -\frac{1}{\sqrt{2}}\left(S^3+iS^0\right)_{\tilde{a}\tilde{b}\tilde{c}}\,, \\
d^{11}_{\bar{a}\bar{b}} & \equiv \frac{1}{\sqrt{2}}\left(d^1-id^2\right)_{\bar{a}\bar{b}}\,, \qquad d^{12}_{\bar{a}\bar{b}} \equiv -\frac{1}{\sqrt{2}}\left(d^3-id^0\right)_{\bar{a}\bar{b}}\,, \\
d^{22}_{\bar{a}\bar{b}} & \equiv \frac{1}{\sqrt{2}}\left(d^1+id^2\right)_{\bar{a}\bar{b}}\,, \qquad d^{21}_{\bar{a}\bar{b}} \equiv -\frac{1}{\sqrt{2}}\left(d^3+id^0\right)_{\bar{a}\bar{b}}\,.
\end{aligned}
\tag{4.23}
$$

From the explicit forms in (4.22), one can calculate the OPEs between them and arrives at the following first order pole as follows with (4.23):

$$
\begin{aligned}
\mathbf{G}_{\alpha\beta}(z)\, G'_{\rho\sigma}(w)|_{\frac{1}{(z-w)}} & = \frac{1}{(k+N+2)^2}\Big[(k+N+2)h^{\alpha\beta}_{\bar{a}\bar{b}}\, d^{\rho\sigma\bar{a}}{}_{\bar{d}} V^{\bar{b}} V^{\bar{d}} - k h^{\alpha\beta}_{\bar{a}\bar{b}}\, d^{\rho\sigma\bar{b}}{}_{\bar{c}}\partial Q^{\bar{a}} Q^{\bar{c}} \\
& + h^{\alpha\beta}_{\tilde{a}\tilde{b}}\, d^{\rho\sigma}_{\bar{c}\bar{d}}\, f^{\tilde{b}\bar{d}}{}_{e} Q^{\tilde{a}} Q^{\bar{c}} V^{e} - \frac{1}{2} S^{\alpha\beta}_{\bar{a}\tilde{b}\tilde{c}}\, d^{\rho\sigma\bar{a}}{}_{\bar{e}} Q^{\tilde{b}} Q^{\tilde{c}} V^{\bar{e}}\Big](w).
\end{aligned}
\tag{4.24}
$$

Then by taking the appropriate six cases in (4.24), one obtains the final six higher spin-2 currents together with the derivative term of higher spin-1 current.

For the higher spin-$\frac{5}{2}$ currents, one should calculate the OPE between the spin-$\frac{3}{2}$ currents and the higher spin-2 currents obtained in previous paragraph. Then it is straightforward to calculate the OPEs between the spin-$\frac{3}{2}$ currents and (4.24). Once the higher spin-$\frac{5}{2}$ currents are obtained, then one should repeat the above procedure. That is, in order to obtain the final higher spin-3 current, the OPEs between the spin-$\frac{3}{2}$ currents and the first order poles where the composite fields have the conformal dimension $\frac{5}{2}$ are needed. Although the complete expressions are not written down in this paper, one can follow the procedures in the nonlinear version [2]. See also Appendix F which contains the precise relations between the higher spin currents in the nonlinear and linear versions for general N and k.

5 Three-point functions in an extension of large $\mathcal{N}=4$ linear superconformal algebra

As in section 3, one calculates the three point functions for the higher spin currents in the extension of large $\mathcal{N}=4$ linear superconformal algebra.

5.1 Eigenvalue equations for spin-2 current acting on the states $|(f;0)>$ and $|(0;f)>$ in the large $\mathcal{N}=4$ linear superconformal algebra

Let us define the $\mathbf{U}$-charge as in [6].

$$\begin{aligned} i\mathbf{U}_0|(f;0)> &= \mathbf{u}(f;0)|(f;0)>, \\ i\mathbf{U}_0|(0;f)> &= \mathbf{u}(0;f)|(0;f)>. \end{aligned} \tag{5.1}$$

From the explicit expression in (4.13), one can obtain the eigenvalues $\mathbf{u}(f;0)$ and $\mathbf{u}(0;f)$ as follows:

$$\begin{aligned} \mathbf{u}(f;0) &= -\frac{1}{2}\sqrt{\frac{N}{N+2}}, \\ \mathbf{u}(0;f) &= \frac{1}{2}\sqrt{\frac{N+2}{N}}. \end{aligned} \tag{5.2}$$

One has the explicit expressions [26].

From the spin-2 current in (F.1), one has the following relation

$$\begin{aligned} \mathbf{T}_0|(f;0)> &\sim \left[T - \frac{1}{(k+N+2)}\mathbf{U}\mathbf{U}\right]_0 |(f;0)> \\ &= \left[h(f;0) + \frac{1}{(k+N+2)}\mathbf{u}^2(f;0)\right]|(f;0)> \\ &= \left[\frac{(N+1)(N+3)}{2(N+2)(k+N+2)}\right]|(f;0)>. \end{aligned} \tag{5.3}$$

In the first line, the spin-$\frac{1}{2}$ current dependent terms are ignored. See also (4.12) where $\mathbf{\Gamma}^{\mu}$ term contains the spin-$\frac{1}{2}$ current. In the second and third lines, the relations (3.5), (5.1) and

[26] For $N=3$, one considers the following matrix acting on the states

$$i\mathbf{U}_0|(f;\star)> = \left(\begin{array}{ccc|cc} \frac{1}{\sqrt{15}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{15}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{15}} & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{3}{5}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{3}{5}} \end{array}\right)|(f;\star)>.$$

Then one can vary N for $N=5,7,9$ as before and one sees the general N behavior as in (5.2). One can show that the spin-1 current $\mathbf{U}(z)$ is equivalent to the previous spin-1 current $U(z)$ in the footnote 7. In other words, $\mathbf{U}(z) = -\frac{i}{2\sqrt{N(N+2)}}U(z)$. From the explicit form for the spin-1 current, the relevant term can be described as $i\mathbf{U}(z) \sim \frac{1}{2(5+k)}\sqrt{\frac{5}{3}}\sum_{a=1}^{6} Q^a Q^{a^*}(z)$. Then one can calculate the following OPE $i\mathbf{U}(z)\,Q^{\bar{A}^*}(w) = \frac{1}{(z-w)}\left[\frac{1}{2}\sqrt{\frac{5}{3}}\right]Q^{\bar{A}^*}(w)+\cdots$. Therefore, the eigenvalue is given by $\frac{1}{2}\sqrt{\frac{5}{3}}$ for $N=3$.

(5.2) are used. The eigenvalue in the equation (5.3) is the same value as $h'(f;0)$ appeared in [1]. One can see this by writing $\frac{(N+1)(N+3)}{2(N+2)}$ as $\left[\frac{(N+2)}{2}-\frac{1}{2(N+2)}\right]$ which is nothing but the quadratic Casimir of fundamental representation in $SU(N+2)$. Therefore, the $\hat{u}$ part and u part of [1] in the conformal dimension are canceled each other completely.

Because the OPE between $\mathbf{\Gamma}^{\mu}(z)$ and $Q^{\bar{A}^*}(w)$ is regular, $\partial\mathbf{\Gamma}^{\mu}\mathbf{\Gamma}_{\mu}(z)$ term does not contribute to the eigenvalue equation. Then one obtains the zero mode eigenvalue equation of $\mathbf{T}(z)$ for the state $|(0;f)>$ as follows [27]:

$$\begin{aligned}
\mathbf{T}_0|(0;f)> &\sim \left[T-\frac{1}{(k+N+2)}\mathbf{U}\mathbf{U}\right]_0|(0;f)> \\
&= \left[h(0;f)+\frac{1}{(k+N+2)}\mathbf{u}^2(0;f)\right]|(0;f)> \\
&= \left[\frac{(Nk+2N+1)}{2N(k+N+2)}\right]|(0;f)> . \qquad (5.4)
\end{aligned}$$

In the third line, the relation (3.9) and (5.2) are used. The eigenvalue in the equation (5.4) is exactly the same as $h'(0;f)$ described in [1]. One can also understand this by writing the eigenvalue as $\frac{1}{2}-\left[\frac{(\frac{N}{2}-\frac{1}{2N})}{(k+N+2)}\right]$ where the second term is a quadratic Casimir in the fundamental representation in $SU(N)$ and the first term $\frac{1}{2}$ is the conformal dimension from an excitation number. The $\hat{u}$ part and u part of [1] in the conformal dimension are canceled each other completely.

The large N limit (3.1) for (5.3) and (5.4) leads to

$$\begin{aligned}
\mathbf{T}_0|(f;0)> &= \frac{1}{2}\lambda|(f;0)>, \\
\mathbf{T}_0|(0;f)> &= \frac{1}{2}(1-\lambda)|(0;f)> . \qquad (5.5)
\end{aligned}$$

which are exactly the same as the ones in the subsection 3.1 in the nonlinear version. In other words, the extra $\mathbf{U}\mathbf{U}$ term in (5.3) and (5.4) does not contribute to the eigenvalue equation in this large N limit.

[27] One determines the zero mode eigenvalue for the 'light' state as follows:

$$\begin{aligned}
\mathbf{T}_0|(f;f)> &\sim \left[T-\frac{1}{(k+N+2)}\mathbf{U}\mathbf{U}\right]_0|(f;f)>=\left[h(f;f)+\frac{1}{(k+N+2)}\mathbf{u}^2(f;f)\right]|(f;f)> \\
&= \left[\frac{(N+1)^2}{N(N+2)(k+N+2)}\right]|(f;f)>\rightarrow\frac{\lambda}{N}|(f;f)>\rightarrow 0,
\end{aligned}$$

where we used $\mathbf{u}(f;f)=\frac{1}{\sqrt{N(N+2)}}$. See also the footnote 26 for the 3×3 diagonal elements, $\frac{1}{\sqrt{15}}$, when $N=3$. In the last line, the large N limit (3.1) is taken.

5.2 Eigenvalue equations for higher spin-1 current acting on the states $|(f;0)>$ and $|(0;f)>$

In this case, the previous relations (3.12) and (3.13) hold because of (4.16).

5.3 Eigenvalue equations for higher spin currents of spins 2 and 3 acting on the states $|(f;0)>$ and $|(0;f)>$

Let us calculate the eigenvalue equation for the higher spin-2 current. It turns out that one obtains the following result

$$\begin{aligned}
\mathbf{T}_0^{(2)}|(f;0)>_{\pm} &= \pm\left[\frac{N}{2(N+k+2)}\right]|(f;0)>_{\pm}, \\
\mathbf{T}_0^{(2)}|(0;f)>_{\pm} &= \pm\left[\frac{k}{2(N+k+2)}\right]|(0;f)>_{\pm}\,.
\end{aligned} \tag{5.6}$$

Compared to the previous expression (3.17) in the nonlinear version, the above eigenvalues (5.6) appear in the factors in (3.17). The extra terms in the higher spin-2 current in the linear version contribute to the eigenvalue equation also and can be added to the right hand side of (3.17). Then the above simple result occurs. Furthermore, the previous relation (3.18) holds in this case. More explicitly, the expressions $\frac{(N-2k)}{(2Nk+N+k)}$ and $-\frac{(N-2k)}{(2Nk+N+k)}$ (with an overall factor) in the eigenvalues of first two terms in (3.17) arise from the eigenvalue equations for the zero mode of the extra terms in the fifth equation of Appendix (F.2). Similarly, the expressions $\frac{(k-2N)}{(2Nk+N+k)}$ and $-\frac{(k-2N)}{(2Nk+N+k)}$ in the eigenvalues of the last two terms in (3.17) can be analyzed.

For the other higher spin-2 current, the similar calculation gives the following result

$$\begin{aligned}
\mathbf{W}_0^{(2)}|(f;0)>_{\pm} &= \mp\left[\frac{N}{2(N+k+2)}\right]|(f;0)>_{\pm}, \\
\mathbf{W}_0^{(2)}|(0;f)>_{\pm} &= \pm\left[\frac{k}{2(N+k+2)}\right]|(0;f)>_{\pm}\,.
\end{aligned} \tag{5.7}$$

The eigenvalue equations (5.7) look similar to the previous ones in (5.6) up to the signs. Furthermore, compared to the previous ones in the nonlinear version, one sees the eigenvalues in the factors of (3.20). In this case also, the extra terms in the higher spin-2 current in the linear version contribute to the eigenvalue equation also and can be added to the right hand side of (3.20). Then the above very simple result can be obtained. Simple linear combinations between these higher spin-2 currents will give rise to simple eigenvalue equations which will appear in next subsection. One sees the previous relation (3.21) in this case. The expressions $-\frac{(N-2k)}{(2Nk+N+k)}$ and $\frac{(N-2k)}{(2Nk+N+k)}$ in the eigenvalues of the first two equations in (3.20) can be

analyzed from the last equation of Appendix (F.2). Also the expressions $\frac{(k-2N)}{(2Nk+N+k)}$ and $-\frac{(-2N+k)}{(2Nk+N+k)}$ appearing in the eigenvalues of the last two equations in (3.20) can be described similarly.

Furthermore, if one replaces the fundamental representation f with the antifundamental representation $\bar{f}$ in (5.6) and (5.7), then the right hand sides remain unchanged.

By taking the large N 't Hooft limit (3.1), one obtains

$$\begin{aligned}
\mathbf{T}_0^{(2)}|(f;0)>_{\pm} &= \pm\frac{1}{2}\lambda|(f;0)>_{\pm}, \\
\mathbf{T}_0^{(2)}|(0;f)>_{\pm} &= \pm\frac{1}{2}(1-\lambda)|(0;f)>_{\pm}, \\
\mathbf{W}_0^{(2)}|(f;0)>_{\pm} &= \mp\frac{1}{2}\lambda|(f;0)>_{\pm}, \\
\mathbf{W}_0^{(2)}|(0;f)>_{\pm} &= \pm\frac{1}{2}(1-\lambda)|(0;f)>_{\pm}\,.
\end{aligned} \tag{5.8}$$

Surprisingly, these eigenvalue equations (5.8) are exactly the same as the previous ones in the nonlinear version (3.23).

For the final higher spin-3 current [28], the following eigenvalue equations hold

$$\begin{aligned}
\mathbf{W}_0^{(3)}|(f;0)> &= \mathbf{w^{(3)}}(f;0)|(f;0)>, \\
\mathbf{W}_0^{(3)}|(0;f)> &= \mathbf{w^{(3)}}(0;f)|(0;f)>, \\
\mathbf{w^{(3)}}(f;0) &\equiv \frac{2N}{3(N+2)(N+k+2)^2(4N+5+(3N+4)k)} \times \Big[(5N^3+26N^2+50N+30) \\
&+ (6N^3+36N^2+71N+43)k+(3N^2+10N+8)k^2\Big], \\
\mathbf{w^{(3)}}(0;f) &\equiv -\frac{2k\left[(4N^3+23N^2+18N)+(3N^3+24N^2+16N-3)k+(6N^2+5N)k^2\right]}{3N(N+k+2)^2(4N+5+(3N+4)k)}.
\end{aligned} \tag{5.9}$$

Based on the partial result in the footnote 28, one can further calculate those quantities for $N=5,7,9$. It is not difficult to obtain the denominators in 44 and 55 elements for generic N. It is nontrivial to see the generic N behavior for the numerators. The numerator is a quadratic in k and one can introduce the three independent polynomials in N with the highest power 3 (with four unknown coefficients) in the quadratic in k, in the linear in k and in the constant terms respectively. Now one can use the four constraint equations from the above $N=3,5,7,9$ cases. It turns out the above unknown coefficients are completely determined uniquely. For the eigenvalue $\mathbf{w^{(3)}}(0;f)$, one can analyze similarly. The numerator behaves nontrivially. So one introduces three polynomials of order 3 with undetermined coefficients. They can be fixed by solving the relevant equations for $N=3,5,7,9$.

[28] Explicitly one obtains the first three diagonal matrix elements $-\frac{52(k-3)(5k+9)}{15(k+5)^2(13k+17)}$, and the remaining two diagonal matrix elements $\frac{2\left(65k^2+742k+549\right)}{5(k+5)^2(13k+17)}$ for $N=3$.

Furthermore, if one replaces the fundamental representation f with the antifundamental representation $\bar{f}$ in (5.9), then the right hand sides have minus signs.

It is obvious that there is no $N \leftrightarrow k$ symmetry which is a different aspect compared to the $W_0^{(3)}$ case in the nonlinear version. However, under the large N 't Hooft limit (3.1), the above expressions (5.9) become simple form as follows:

$$\begin{aligned} \mathbf{W}_0^{(3)}|(f;0)> &= \frac{2}{3}\lambda(1+\lambda)|(f;0)>, \\ \mathbf{W}_0^{(3)}|(0;f)> &= -\frac{2}{3}(1-\lambda)(2-\lambda)|(0;f)> . \end{aligned} \tag{5.10}$$

This is exactly the same as the ones in (3.27) for the $W_0^{(3)}$ eigenvalue equations in the nonlinear version [29].

5.4 Eigenvalue equations for higher spin currents of spins 2 and 3 acting on the states $|(f;0)>$ and $|(0;f)>$ in the basis of [24]

The eigenvalue equations in the nonlinear version [24] are the same as the ones in the linear version. From the explicit relations in [27],

$$\begin{aligned} V_1^{(1),\pm 1}(z) &= 2i\left(\mathbf{U}_{\mp}^{(2)} - \mathbf{V}_{\pm}^{(2)}\right)(z), \\ V_1^{(1),\pm 2}(z) &= -2\left(\mathbf{U}_{\mp}^{(2)} + \mathbf{V}_{\pm}^{(2)}\right)(z), \\ V_1^{(1),\pm 3}(z) &= \pm 2i\left(\mathbf{T}^{(2)} \mp \mathbf{W}^{(2)}\right)(z), \end{aligned} \tag{5.11}$$

one can rewrite the eigenvalue equations as follows:

$$\begin{aligned} \left[V_1^{(1),+3}\right]_0 |(f;0)>_{\pm} &= \pm\left[\frac{2iN}{(N+k+2)}\right]|(f;0)>_{\pm}, \\ \left[V_1^{(1),+3}\right]_0 |(0;f)> &= 0, \\ \left[V_1^{(1),-3}\right]_0 |(f;0)> &= 0, \\ \left[V_1^{(1),-3}\right]_0 |(0;f)>_{\pm} &= \mp\left[\frac{2ik}{(N+k+2)}\right]|(0;f)>_{\pm} . \end{aligned} \tag{5.12}$$

---

[29] One also calculates the eigenvalue equation for the 'light' state as follows:

$$\mathbf{W}_0^{(3)}|(f;f)>= \left[\frac{4(N-k)(4N^3+19N^2+22N+6+(3N^3+10N^2+8N)k)}{3N(N+2)(N+k+2)^2(4N+5+(3N+4)k)}\right]|(f;f)>\rightarrow \frac{4\lambda(2\lambda-1)}{3N}|(f;f)>\rightarrow 0.$$

As in the footnote 28, the first three elements in $N=3$ case are relevant to this expression. By following the prescription in (5.9), the explicit N dependence can be fixed completely for $N=3,5,7,9$. Note that there are two cubic polynomials in N in the numerator.

The corresponding large N 't Hooft limit (3.1) in (5.12) provides the following result,

$$\begin{aligned}
\left[V_1^{(1),+3}\right]_0 |(f;0)>_\pm &= \pm 2i\lambda |(f;0)>_\pm, \\
\left[V_1^{(1),+3}\right]_0 |(0;f)> &= 0, \\
\left[V_1^{(1),-3}\right]_0 |(f;0)> &= 0, \\
\left[V_1^{(1),-3}\right]_0 |(0;f)>_\pm &= \mp 2i(1-\lambda)|(0;f)>_\pm .
\end{aligned} \tag{5.13}$$

In this case (5.13), the eigenvalue equation contains zero values and one cannot take one of the ratios between them.

By introducing the following quantities

$$\begin{aligned}
V_1^{(1)+\pm}(z) &\equiv \left[V_1^{(1)+1} \pm i V_1^{(1)+2}\right](z), \\
V_1^{(1)-\pm}(z) &\equiv \left[V_1^{(1)-1} \pm i V_1^{(1)-2}\right](z),
\end{aligned} \tag{5.14}$$

one can calculate [30]

$$\begin{aligned}
\left[V_1^{(1)+\pm}\right]_0 |(f;0)>_\mp &= -\left[\frac{4Ni}{(N+k+2)}\right] |(f;0)>_\pm \rightarrow -4i\lambda |(f;0)>_\pm, \\
\left[V_1^{(1)-\pm}\right]_0 |(0;f)>_\mp &= \left[\frac{4ki}{(N+k+2)}\right] |(0;f)>_\pm \rightarrow 4i(1-\lambda)|(0;f)>_\pm .
\end{aligned} \tag{5.15}$$

The quadratic combinations will be obtained later [31].

Furthermore, there exists the following relation [27] which was found for $N=3$, together with (4.6) and (4.16)

$$\begin{aligned}
V_2^{(1)}(z) &= 4\left[\mathbf{W^{(3)}} + \frac{4(k-N)}{(4N+5+(3N+4)k)}\left(\mathbf{T}\,\mathbf{T^{(1)}} - \frac{1}{2}\partial^2 \mathbf{T^{(1)}}\right)\right](z) \\
&= 4\left[\mathbf{W^{(3)}} + \frac{4(k-N)}{(4N+5+(3N+4)k)}\mathbf{T^{(1)}}\,\mathbf{T}\right](z).
\end{aligned} \tag{5.16}$$

---

[30] More precisely, one has

$$\begin{aligned}
\left[V_1^{(1)-+}\right]_0 \frac{1}{\sqrt{k+N+2}} Q_{-\frac{1}{2}}^{(a+N)^*}|0> &= \left[\frac{4ki}{(N+k+2)}\right] \frac{1}{\sqrt{k+N+2}} Q_{-\frac{1}{2}}^{a^*}|0>, \\
\left[V_1^{(1)--}\right]_0 \frac{1}{\sqrt{k+N+2}} Q_{-\frac{1}{2}}^{a^*}|0> &= \left[\frac{4ki}{(N+k+2)}\right] \frac{1}{\sqrt{k+N+2}} Q_{-\frac{1}{2}}^{(a+N)^*}|0>,
\end{aligned}$$

where $a = 1, 2, \cdots, N$.

[31] One can also present the three point functions together with (5.14) and (5.15) as follows:

$$\begin{aligned}
< \overline{\mathcal{O}}_{+,\pm}\mathcal{O}_{+,\mp} V_1^{(1)+\pm} > &= -4i\lambda < \overline{\mathcal{O}}_{+,\pm}\mathcal{O}_{+,\pm} >, \\
< \overline{\mathcal{O}}_{-,\pm}\mathcal{O}_{-,\mp} V_1^{(1)-\pm} > &= 4i(1-\lambda) < \overline{\mathcal{O}}_{-,\pm}\mathcal{O}_{-,\pm} > .
\end{aligned}$$

This holds for any value of N which is obtained by varying the N values. Note that the N, k dependence in the second term is very simple. In the last line of (5.16), the derivative term by changing the commutator [9] in $\left[\mathbf{T^{(1)}},\, \mathbf{T}\right](z) = -\frac{1}{2}\partial^2 \mathbf{T^{(1)}}(z)$ is used in order to simplify the zero mode eigenvalue equation. For $N = k$, the above spin-3 current $V_2^{(1)}(z)$ becomes a primary current under the stress energy tensor (4.6).

It turns out that one obtains very simple eigenvalue equations

$$\begin{aligned}
\left[V_2^{(1)}\right]_0 |(f;0)> &= \left[\frac{8N(2N+k+3)}{3(N+k+2)^2}\right] |(f;0)>, \\
\left[V_2^{(1)}\right]_0 |(0;f)> &= -\left[\frac{8k(2k+N+3)}{3(N+k+2)^2}\right] |(0;f)> . \qquad (5.17)
\end{aligned}$$

Note that one can also check the correctness of (5.9) by looking at (5.17) and (5.16). The reason is as follows: Because the zero mode of $\mathbf{T^{(1)}T}$ acting on the two states are known for general N and k. The coefficient in the second term of (5.16) depends on N, k explicitly. By combining these two contributions, one obtains (5.9) exactly.

From this (5.17), one has the following relation

$$\left(\left[V_2^{(1)}\right]_0 |(f;0)>\right)_{N\leftrightarrow k, 0\leftrightarrow f} = -\left[V_2^{(1)}\right]_0 |(0;f)> . \qquad (5.18)$$

Therefore, in this particular basis, there exists a $N \leftrightarrow k$ symmetry (5.18) up to signs.

The large N 't Hooft limit (3.1) in (5.17) leads to the following result

$$\begin{aligned}
\left[V_2^{(1)}\right]_0 |(f;0)> &= \frac{8}{3}\lambda(1+\lambda)|(f;0)>, \\
\left[V_2^{(1)}\right]_0 |(0;f)> &= -\frac{8}{3}(1-\lambda)(2-\lambda)|(0;f)> . \qquad (5.19)
\end{aligned}$$

Note the four times eigenvalues of $\mathbf{W}_0^{(3)}$ are the same as those of $V_2^{(1)}$ in the large N 't Hooft limit (5.19). The second term in the bracket of (5.16) does not contribute to the eigenvalue equation in the large N 't Hooft limit [32].

One can also calculate the eigenvalue equations for the sum of the square of spin-2 currents acting on the two states with (5.11) and it turns out that

$$\left[\sum_{i=1}^{3} V_1^{(1)+i} V_1^{(1)+i}\right]_0 |(f;0)> = -\left[\frac{12N(5N+4k+4)}{(N+k+2)^2}\right] |(f;0)>,$$

[32] One also sees the following result

$$\left[V_2^{(1)}\right]_0 |(f;f)> = \left[\frac{16(N-k)}{3(N+k+2)^2}\right] |(f;f)> \rightarrow \frac{16\lambda(2\lambda-1)}{3N} |(f;f)> \rightarrow 0.$$

$$\left[\sum_{i=1}^{3} V_1^{(1)+i} V_1^{(1)+i}\right]_0 |(0;f)> \;=\; -\left[\frac{48k}{(N+k+2)^2}\right]|(0;f)>,$$
$$\left[\sum_{i=1}^{3} V_1^{(1)-i} V_1^{(1)-i}\right]_0 |(f;0)> \;=\; -\left[\frac{48N}{(N+k+2)^2}\right]|(f;0)>,$$
$$\left[\sum_{i=1}^{3} V_1^{(1)-i} V_1^{(1)-i}\right]_0 |(0;f)> \;=\; -\left[\frac{12k(5k+4N+4)}{(N+k+2)^2}\right]|(0;f)>. \tag{5.20}$$

One can interpret the eigenvalues, $-\frac{12N\times 2}{(N+k+2)^2}$, $-\frac{24k}{(N+k+2)^2}$, $-\frac{24N}{(N+k+2)^2}$ and $\frac{-12k\times 2}{(N+k+2)^2}$ appearing in the right hand side of (5.20) come from the extra terms between the left hand side of (3.29) and the left hand side of (5.20). There is a $N \leftrightarrow k$ symmetry between the first and the last (and also the second and the third) in (5.20). One also has the large N 't Hooft limits as follows: $-12\lambda(4+\lambda)$, $-\frac{48\lambda(1-\lambda)}{N}$, $-\frac{48\lambda^2}{N}$ and $-12(1-\lambda)(5-\lambda)$ respectively. In the footnote 8, the similar calculations were done for the spin-1 currents. Here the conformal dimensions of $V_1^{(1)\pm i}(z)$ are given by two and the sum over $SU(2)$ indices in the quadratic of the higher spin-2 currents is taken. The zero mode eigenvalue equations depend on N and k. It would be interesting to study the representation theory concerning on the higher spin currents further by generalizing the previous works in [6]. Under the large N 't Hooft limit, the nonzero eigenvalue corresponding to the quadratic higher spin-2 currents $V_1^{(1),+i}$ appears in the state $|(f;0)>$ while the nonzero eigenvalue corresponding to the quadratic higher spin-2 currents $V_1^{(1),-i}$ appears in the state $|(0;f)>$. The corresponding three point functions can be described without any difficulty [33].

From the eigenvalues

$$\mathbf{u}(0;[2,0,\cdots,0]) = \mathbf{u}(0;[0,1,0,\cdots,0]) = \sqrt{\frac{N+2}{N}}, \tag{5.21}$$

which can be obtained from (3.31) and the spin-1 current $\mathbf{U}(z)$, one can find the conformal dimensions in the linear version together with (5.21)

$$h'(0;[2,0,\cdots,0]) \;=\; h(0;[2,0,\cdots,0]) + \frac{1}{(k+N+2)}\mathbf{u}^2(0;[2,0,\cdots,0])$$
$$=\; \frac{(Nk+N+2)}{N(k+N+2)},$$

[33] For the light state, one obtains $\left[\sum_{i=1}^{3} V_1^{(1)+i} V_1^{(1)+i}\right]_0 |(f;f)>= -\left[\frac{48(2k+1)}{(N+k+2)^2}\right]|(f;f)>\rightarrow -\frac{96\lambda(1-\lambda)}{N}|(f;f)>$ and $\left[\sum_{i=1}^{3} V_1^{(1)-i} V_1^{(1)-i}\right]_0 |(f;f)>= -\left[\frac{48(2N+1)}{(N+k+2)^2}\right]|(f;f)>\rightarrow -\frac{96\lambda^2}{N}|(f;f)>$ and the corresponding relations in the nonlinear version are given by $\left[\sum_{i=1}^{3} \tilde{V}_1^{(1)+i} \tilde{V}_1^{(1)+i}\right]_0 |(f;f)>= -\left[\frac{96k}{(N+k+2)^2}\right]|(f;f)>\rightarrow -\frac{96\lambda(1-\lambda)}{N}|(f;f)>$ and $\left[\sum_{i=1}^{3} \tilde{V}_1^{(1)-i} \tilde{V}_1^{(1)-i}\right]_0 |(f;f)>= -\left[\frac{96N}{(N+k+2)^2}\right]|(f;f)>\rightarrow -\frac{96\lambda^2}{N}|(f;f)>$. All of these go to zero under the large N 't Hooft limit.

$$
\begin{aligned}
h'(0;[0,1,0,\cdots,0]) &= h(0;[0,1,0,\cdots,0]) + \frac{1}{(k+N+2)}\mathbf{u}^2(0;[0,1,0,\cdots,0]) \\
&= \frac{(Nk+3N+2)}{N(k+N+2)}.
\end{aligned} \tag{5.22}
$$

Note that in (5.22), the differences between the conformal dimensions in the nonlinear and linear versions can be seen from $(C.8)$ and $(C.9)$ of [1]. Similarly, from the state (3.33), one obtains the following eigenvalue

$$
\mathbf{u}(0;[0^{p-1},1,0,\cdots,0]) = \frac{p}{2}\sqrt{\frac{N+2}{N}}, \tag{5.23}
$$

and the following conformal dimension together with (5.23) can be obtained

$$
\begin{aligned}
h'(0;[0^{p-1},1,0,\cdots,0]) &= h(0;[0^{p-1},1,0,\cdots,0]) + \frac{1}{(k+N+2)}\mathbf{u}^2(0;[0^{p-1},1,0,\cdots,0]) \\
&= \frac{p(Np+Nk+N+p)}{2N(k+N+2)}.
\end{aligned} \tag{5.24}
$$

Also in this case (5.24), the difference between the conformal dimensions can be seen from the last equation of [1] [34].

Therefore, in this section, the three point functions can be summarized by (5.5), (5.8) and (5.10). As in the nonlinear version one obtains the following ratios for the three point functions

$$
\begin{aligned}
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+\mathbf{T^{(1)}}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-\mathbf{T^{(1)}}>} &= \left[\frac{\lambda}{1-\lambda}\right], \\
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+\mathbf{T^{(2)}}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-\mathbf{T^{(2)}}>} &= \left[\frac{\lambda}{1-\lambda}\right], \\
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+\mathbf{W^{(2)}}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-\mathbf{W^{(2)}}>} &= -\left[\frac{\lambda}{1-\lambda}\right], \\
\frac{<\overline{\mathcal{O}}_+\mathcal{O}_+\mathbf{W^{(3)}}>}{<\overline{\mathcal{O}}_-\mathcal{O}_-\mathbf{W^{(3)}}>} &= -\left[\frac{\lambda(1+\lambda)}{(1-\lambda)(2-\lambda)}\right].
\end{aligned}
$$

These are exactly the same as the ones in (3.28). Under the large N 't Hooft limit, the ratios of the three point functions in the nonlinear and linear versions are equivalent to each other.

[34] From the relation

$$
\begin{aligned}
[-\mathbf{UU}]_0\,|(f;\bar{f})> &= \left[\mathbf{u}^2(f;0)+\mathbf{u}^2(0;\bar{f})+\mathrm{diag}\left(-\frac{1}{N},\cdots,-\frac{1}{N},\frac{1}{2},\frac{1}{2}\right)\right]|(f;\bar{f})> \\
&= \left[\frac{(N+1)^2}{N(N+2)}\right]|(f;\bar{f})>,
\end{aligned}
$$

where the eigenvalue $\mathbf{u}(0;\bar{f}) = -\frac{1}{2}\sqrt{\frac{N+2}{N}}$ is used, one obtains the following conformal dimension $h'(f;\bar{f}) = h(f;\bar{f}) + \frac{1}{(k+N+2)}\frac{(N+1)^2}{N(N+2)} = \frac{1}{2} + \frac{(N+1)^2}{N(N+2)(N+k+2)}$. The previous result (3.37) is substituted.

6 Conclusions and outlook

In this paper, the three point functions in the large $\mathcal{N}=4$ holography in the large N 't Hooft limit are obtained. In the three point functions, scalar $-$ $\overline{\text{scalar}}$ $-$ current, the two scalars are characterized by the coset primaries corresponding to the two states $|(0;f)>$ and $|(f;0)>$ in the minimal representations of the coset. The currents are given by the spin-1 currents, spin-2 current of the large $\mathcal{N}=4$ (non)linear superconformal algebra, the higher spin-1 current, the higher spin-2 currents and the higher spin-3 current.

• Three point functions in the bulk

As in [23], it is an open problem to obtain the asymptotic symmetry algebra of the higher spin theory on the AdS_3 space. Once this is found explicitly, then one can compare the results of this paper with the corresponding three point functions which can be obtained indirectly in the bulk.

• The general s dependence of three point function

From the three point functions with spins $2,3$, one can expect that the ratio of three point function for given spin s looks like as $\frac{\lambda(1+\lambda)\cdots(s-2+\lambda)}{(1-\lambda)(2-\lambda)\cdots(s-1-\lambda)}$ up to signs. It is an open problem to see whether this is general behavior or not. For the higher spin-4 current, one can apply the present method to the orthogonal coset theory where one has the higher spin-4 current in the lowest $\mathcal{N}=4$ higher spin current. It is a good exercise to see whether one sees the above three point function with $s=4$ and determines whether the behavior looks like the one in [28].

• An extension of small $\mathcal{N}=4$ linear superconformal algebra

As described in the introduction, the small $\mathcal{N}=4$ linear superconformal algebra can be obtained by taking the large level limit in the large $\mathcal{N}=4$ linear superconformal algebra. Then one can obtain an extension of the small $\mathcal{N}=4$ linear superconformal algebra from the extension of large $\mathcal{N}=4$ linear superconformal algebra. Therefore, in this construction, the complete OPEs between the 16 currents of large $\mathcal{N}=4$ linear superconformal algebra and the 16 lowest higher spin currents for general N and k should be obtained.

• Oscillator formalism for the higher spin currents

According to the original Vasiliev's oscillator formalism, some part of calculations in the higher spin currents of large $\mathcal{N}=4$ nonlinear superconformal algebra are obtained in [1]. It is an open problem to see whether one can see the oscillator formalism in an extension of the large $\mathcal{N}=4$ linear superconformal algebra.

• The operator product expansion of the 16 higher spin currents in $\mathcal{N}=4$ superspace

So far, the complete OPEs between the 16 currents and the 16 higher spin currents for

general N and k are not known although its nonlinear version appears in [24] where the coset field realizations are not checked. The first step is to write down the complete OPEs in an extension of large $\mathcal{N}=4$ linear superconformal algebra in the $\mathcal{N}=4$ superspace because it is more plausible to consider the linear version rather than the nonlinear version.

• Three point functions in the coset theory which contains an orthogonal Wolf space

One can also consider the different large $\mathcal{N}=4$ holography based on the orthogonal Wolf space [29]. The nonlinear version contains the Wolf space $\frac{SO(N+4)}{SO(N)\times SU(2)\times SU(2)}$ while the linear version contains the coset $\frac{SO(N+4)}{SO(N)\times SU(2)}\times U(1)$. In this case, the minimal representations contain the two states $|(0;v)>$ and $|(v;0)>$ where the former is the vector representation in the $SO(N)$ among the singlets in the $SO(N+4)$ and the latter is the vector representation in the $SO(N+4)$ and at the same time is the singlet under the $SO(N)$. The relevant previous works on this direction are given in [30, 31, 32, 33].

• The next 16 higher spin currents

So far, the higher spin currents in the context of the three point function are the member of the 16 lowest higher spin currents. One can consider the next 16 higher spin currents where the bosonic currents contain the higher spin currents with spins $2,3,4$. One would like to see the behaviors of the three point functions and satisfy whether they behave as above. One expects that as the spin increases, the N dependence for several N in the fractional coefficient functions in the level k becomes complicated. In order to extract the general N behavior, one needs more information about the OPEs for the several N. Furthermore, the basis in [24] is more useful because they already presented the defining OPEs between the 16 currents in the large $\mathcal{N}=4$ linear superconformal algebra and the next 16 higher spin currents.

• Three point functions involving the fermionic (higher spin) currents

In [21], the three point functions which contain the fermionic (higher spin) currents have been described. See also [34]. In this paper, we have only considered the bosonic (higher spin) currents in the three-point functions. It would be interesting to discover the three point functions with fermionic (higher spin) currents explicitly.

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Appendix A The generators of $SU(N+2)$ in complex basis

The $SU(N+2)$ generators can be expressed in the complex basis. There exist $(N+1)$ Cartan generators in $SU(N+2)$ denoted by $H_1, H_2, \cdots, H_{N+1}$ which are defined by [35]

$$[H_m]_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right). \tag{A.1}$$

Then one can define the $\frac{(N+1)}{2}$ diagonal generators with (A.1) as follows:

$$\begin{aligned} T_{p+1} &= iH_1 + H_2, \\ T_{p+2} &= iH_3 + H_4, \\ &\vdots \\ T_{p+\frac{(N+1)}{2}} &= iH_N + H_{N+1}, \end{aligned} \tag{A.2}$$

where one introduces $p \equiv \frac{(N+2)^2-1}{2} - \frac{(N+1)}{2}$. The last generator in (A.2) has a subscript $\frac{(N+2)^2-1}{2}$ which is the last element of adjoint index in complex basis. The remaining $\frac{(N+1)}{2}$ half of diagonal matrix can be obtained by taking the complex conjugation from (A.2).

Among off-diagonal matrices, the $(2N+1)$ matrices have nonzero elements as follows:

$$T_1 = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0\\ & & & & & 0 & 0\\ & & 0 & & & \vdots & \vdots\\ & & & & & 0 & 0\\ & & & & & 0 & 0\\ \hline 1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right), \quad T_2 = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0\\ & & & & & 0 & 0\\ & & 0 & & & \vdots & \vdots\\ & & & & & 0 & 0\\ & & & & & 0 & 0\\ \hline 0 & 1 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right), \cdots$$

$$T_N = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0\\ & & & & & 0 & 0\\ & & 0 & & & \vdots & \vdots\\ & & & & & 0 & 0\\ & & & & & 0 & 0\\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right), \quad T_{N+1} = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0\\ & & & & & 0 & 0\\ & & 0 & & & \vdots & \vdots\\ & & & & & 0 & 0\\ & & & & & 0 & 0\\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right),$$

$$T_{N+2} = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0\\ & & & & & 0 & 0\\ & & 0 & & & \vdots & \vdots\\ & & & & & 0 & 0\\ & & & & & 0 & 0\\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \end{array}\right), \cdots, \quad T_{2N} = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0\\ & & & & & 0 & 0\\ & & 0 & & & \vdots & \vdots\\ & & & & & 0 & 0\\ & & & & & 0 & 0\\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \end{array}\right),$$

$$T_{2N+1} = \left(\begin{array}{ccccc|cc} & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & 0 & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{array}\right). \tag{A.3}$$

The remaining $(2N+1)$ off diagonal matrices can be obtained by taking the transpose for these matrices (A.3). Then the $4N$ coset generators in the nonlinear version are given by (A.3)

$$T_1,\, T_2,\, \cdots,\, T_{2N},\, T_1^{\dagger}(= T_{1^*}),\, T_2^{\dagger}(= T_{2^*}),\, \cdots,\, T_{2N}^{\dagger}(= T_{2N^*}). \tag{A.4}$$

The $(4N+4)$ coset generators in the linear version are given by (A.2) and (A.3)

$$T_1, T_2, \cdots, T_{2N}, T_1^{\dagger}, T_2^{\dagger}, \cdots, T_{2N}^{\dagger}; T_{2N+1}, T_{2N+1}^{\dagger}, T_q, T_q^{\dagger}, \;\; q \equiv p + \frac{(N+1)}{2} = \frac{(N+2)^2-1}{2}. \tag{A.5}$$

That is, compared to (A.4) and (A.5), the extra four generators are given by T_{2N+1} and T_q (and their conjugated ones). Then the remaining $\frac{N(N-1)}{2}$ off-diagonal generators live in the lower half triangle matrix of $N \times N$ matrix. Similarly, the same number of off-diagonal generators live in the upper triangle matrix (by conjugation). Note that the sum of the numbers $N(N-1)$ and $2(2N+1)$ is equal to the difference of $(N+2)^2-1$ and $(N+1)$ as expected.

The metric is

$$g_{ab} = \mathrm{Tr}(T_a T_b) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), a, b = 1, 2, \cdots, \frac{(N+2)^2-1}{2}, 1^*, 2^*, \cdots, (\frac{(N+2)^2-1}{2})^*. \tag{A.6}$$

This is consistent with the description of subsection 2.1. The nonvanishing metric components in (A.6) are

$$g_{AA^*} = g_{A^*A} = 1, \tag{A.7}$$

where $A = 1, 2, \cdots, \frac{(N+2)^2-1}{2}$. For convenience, let us write down the four Cartan generators T_{11} and T_{12} (T_{11^*} and T_{12^*}) in $SU(5)$. One obtains $T_{11} = \mathrm{diag}(\frac{1}{2\sqrt{3}} + \frac{i}{2}, \frac{1}{2\sqrt{3}} - \frac{i}{2}, -\frac{1}{\sqrt{3}}, 0, 0)$ and $T_{12} = \mathrm{diag}(\frac{1}{2\sqrt{10}} + \frac{i}{2\sqrt{6}}, \frac{1}{2\sqrt{10}} + \frac{i}{2\sqrt{6}}, \frac{1}{2\sqrt{10}} + \frac{i}{2\sqrt{6}}, \frac{1}{2\sqrt{10}} - \frac{1}{2}i\sqrt{\frac{3}{2}}, -\sqrt{\frac{2}{5}})$ and their conjugated ones (T_{11^*} and T_{12^*}). The remaining 7 generators (and 7 conjugated ones) can be obtained from (A.3) and the 3 generators (and 3 conjugated ones) live in the 3×3 matrix.

Appendix B The large $\mathcal{N}=4$ nonlinear superconformal algebra

The large $\mathcal{N}=4$ nonlinear superconformal algebra can be summarized by 11 currents as follows: one spin-2 current $T(z)$, four spin-$\frac{3}{2}$ currents $G^{\mu}(z)$, six spin-1 currents $A^{\pm i}(z)$. The explicit OPEs are given by [36]

$$\begin{aligned}
T(z)\,T(w) &= \frac{1}{(z-w)^4}\,\frac{\hat{c}}{2}+\frac{1}{(z-w)^2}\,2T(w)+\frac{1}{(z-w)}\,\partial T(w)+\cdots,\\
T(z)\,\phi(w) &= \frac{1}{(z-w)^2}\,h_{\phi}\,\phi(w)+\frac{1}{(z-w)}\,\partial\phi(w)+\cdots,\\
G^{\mu}(z)\,G^{\nu}(w) &= \frac{1}{(z-w)^3}\,\frac{2}{3}\delta^{\mu\nu}c_{\text{Wolf}}-\frac{1}{(z-w)^2}\,\frac{8}{(k+N+2)}(N\,\alpha^{+i}_{\mu\nu}\,A^{+}_{i}+k\,\alpha^{-i}_{\mu\nu}\,A^{-}_{i})(w)\\
&+ \frac{1}{(z-w)}\,\Big[2\delta^{\mu\nu}T-\frac{4}{(k+N+2)}\partial(N\,\alpha^{+i}_{\mu\nu}\,A^{+}_{i}+k\,\alpha^{-i}_{\mu\nu}\,A^{-}_{i}) \qquad \text{(B.1)}\\
&- \frac{8}{(k+N+2)}(\alpha^{+i}A^{+}_{i}-\alpha^{-i}A^{-}_{i})_{\rho(\mu}(\alpha^{+j}A^{+}_{j}-\alpha^{-j}A^{-}_{j})_{\nu)}{}^{\rho}\Big]\,(w)+\cdots,\\
A^{\pm i}(z)\,G^{\mu}(w) &= \frac{1}{(z-w)}\,\alpha^{\pm i}_{\mu\nu}\,G^{\nu}(w)+\cdots,\\
A^{\pm i}(z)\,A^{\pm j}(w) &= -\frac{1}{(z-w)^2}\,\frac{1}{2}\,\delta^{ij}\,\hat{k}^{\pm}+\frac{1}{(z-w)}\,\epsilon^{ijk}A^{\pm k}(w)+\cdots,
\end{aligned}$$

where two central charges and two levels of $SU(2)_{\hat{k}^+}\times SU(2)_{\hat{k}^-}$ are given by

$$\hat{c}=\frac{3(k+N+2kN)}{(k+N+2)},\quad c_{\text{Wolf}}=\frac{6kN}{(2+k+N)},\quad \hat{k}^{+}=k,\quad \hat{k}^{-}=N. \qquad \text{(B.2)}$$

The conformal dimensions for the currents are $h_{A^{\pm i}}=1$, $h_{G^{\mu}}=\frac{3}{2}$. The quantity $\alpha^{\pm i}$ is defined by

$$\alpha^{\pm i}_{\mu\nu} = \frac{1}{2}\,(\pm\delta_{i\mu}\delta_{\nu 0}\mp\delta_{i\nu}\delta_{\mu 0}+\epsilon_{i\mu\nu})\,.$$

Note that the nonlinear terms in the OPEs between the spin-$\frac{3}{2}$ currents occur. The two levels are given by k and N in (B.2) which are the quantities in the Wolf space coset. The equation (2.6) is an explicit coset realization of the large $\mathcal{N}=4$ nonlinear superconformal algebra.

Appendix C The large $\mathcal{N}=4$ linear superconformal algebra

For convenience, the full expressions for the large $\mathcal{N}=4$ linear superconformal algebra can be summarized by [37]

$$
\begin{aligned}
\mathbf{T}(z)\,\mathbf{T}(w) &= \frac{1}{(z-w)^4}\frac{c}{2}+\frac{1}{(z-w)^2}\,2\mathbf{T}(w)+\frac{1}{(z-w)}\,\partial\mathbf{T}(w)+\cdots, \\
\mathbf{T}(z)\,\phi(w) &= \frac{1}{(z-w)^2}\,h_\phi\,\phi(w)+\frac{1}{(z-w)}\,\partial\phi(w)+\cdots, \\
\mathbf{G}^\mu(z)\,\mathbf{G}^\nu(w) &= \frac{1}{(z-w)^3}\frac{2}{3}\delta^{\mu\nu}c-\frac{1}{(z-w)^2}\frac{8}{(k^++k^-)}(k^-\,\alpha^{+i}_{\mu\nu}\,\mathbf{A}^+_i+k^+\,\alpha^{-i}_{\mu\nu}\,\mathbf{A}^-_i)(w) \\
&+ \frac{1}{(z-w)}\left[2\delta^{\mu\nu}\mathbf{T}-\frac{4}{(k^++k^-)}\partial(k^-\,\alpha^{+i}_{\mu\nu}\,\mathbf{A}^+_i+k^+\,\alpha^{-i}_{\mu\nu}\,\mathbf{A}^-_i)\right](w)+\cdots, \\
\mathbf{A}^{\pm i}(z)\,\mathbf{G}^\mu(w) &= \mp\frac{1}{(z-w)^2}\frac{2k^\pm}{(k^++k^-)}\,\alpha^{\pm i}_{\mu\nu}\,\mathbf{\Gamma}^\nu(w)+\frac{1}{(z-w)}\,\alpha^{\pm i}_{\mu\nu}\,\mathbf{G}^\nu(w)+\cdots, \\
\mathbf{A}^{\pm i}(z)\,\mathbf{A}^{\pm j}(w) &= -\frac{1}{(z-w)^2}\frac{1}{2}k^\pm\delta^{ij}+\frac{1}{(z-w)}\,\epsilon^{ijk}\mathbf{A}^{\pm k}(w)+\cdots, \\
\mathbf{\Gamma}^\mu(z)\,\mathbf{G}^\nu(w) &= \frac{1}{(z-w)}\left[2(\alpha^{+i}_{\mu\nu}\mathbf{A}^+_i-\alpha^{-i}_{\mu\nu}\mathbf{A}^-_i)+\delta^{\mu\nu}\,\mathbf{U}\right](w)+\cdots, \\
\mathbf{A}^{\pm i}(z)\,\mathbf{\Gamma}^\mu(w) &= \frac{1}{(z-w)}\,\alpha^{\pm i}_{\mu\nu}\,\mathbf{\Gamma}^\nu(w)+\cdots, \\
\mathbf{U}(z)\,\mathbf{G}^\mu(w) &= \frac{1}{(z-w)^2}\,\mathbf{\Gamma}^\mu(w)+\cdots, \\
\mathbf{\Gamma}^\mu(z)\,\mathbf{\Gamma}^\nu(w) &= -\frac{1}{(z-w)}\frac{(k^++k^-)}{2}\,\delta^{\mu\nu}+\cdots, \\
\mathbf{U}(z)\,\mathbf{U}(w) &= -\frac{1}{(z-w)^2}\frac{(k^++k^-)}{2}+\cdots,
\end{aligned}
\tag{C.1}
$$

where the indices run over $\mu,\nu=0,1,2,3$ and $i,j,k=1,2,3$. The conformal dimensions are given by $h_{\mathbf{\Gamma}^\mu}=\frac{1}{2}$, $h_{\mathbf{A}^{\pm i}}=1$, $h_{\mathbf{U}}=1$ and $h_{\mathbf{G}^\mu}=\frac{3}{2}$. The levels are given by $k^+=k+1$ and $k^-=N+1$. The central charge is given by $c=\hat{c}+3=\frac{6(k+1)(N+1)}{(N+k+2)}$. The equations (4.6), (4.10), (4.11), (4.12) and (4.13) are explicit coset realization of the large $\mathcal{N}=4$ linear superconformal algebra. Compared to the nonlinear version in Appendix B, there exist four spin-$\frac{1}{2}$ currents $\mathbf{\Gamma}^\mu(z)$ and spin-1 current $\mathbf{U}(z)$. The explicit relations between the 11 currents in the nonlinear and linear versions will appear in Appendix F.

One can introduce the $U(1)$ charge for the $\mathcal{N}=2$ superconformal algebra by reading off the second order pole of $\mathbf{G}^\mu(z)\,\mathbf{G}^\nu(w)$ as follows:

$$
U_{\mathcal{N}=2}(z)\equiv-2i\left(\gamma\mathbf{A}^{+3}+(1-\gamma)\mathbf{A}^{-3}\right)(z),\qquad \gamma\equiv\frac{(N+1)}{(k+N+2)}. \tag{C.2}
$$

Then one obtains the following OPEs with (C.2) where one can read off the corresponding $U(1)$ charges

$$
\begin{aligned}
U_{\mathcal{N}=2}(z)\, Q^{\bar{A}}(w) & = \frac{1}{(z-w)}(1-\gamma)Q^{\bar{A}}(w)+\cdots, (\bar{A}=1,2,\cdots,N), \\
U_{\mathcal{N}=2}(z)\, Q^{\bar{A}}(w) & = -\frac{1}{(z-w)}(1-\gamma)Q^{\bar{A}}(w)+\cdots, (\bar{A}=N+1,N+2,\cdots,2N), \\
U_{\mathcal{N}=2}(z)\, Q^{\bar{A}^*}(w) & = -\frac{1}{(z-w)}(1-\gamma)Q^{\bar{A}^*}(w)+\cdots, (\bar{A}^*=1^*,2^*,\cdots,N^*), \\
U_{\mathcal{N}=2}(z)\, Q^{\bar{A}^*}(w) & = \frac{1}{(z-w)}(1-\gamma)Q^{\bar{A}^*}(w)+\cdots, (\bar{A}^*=(N+1)^*,(N+2)^*,\cdots,2N^*), \\
U_{\mathcal{N}=2}(z)\, V^{\bar{A}}(w) & = \frac{1}{(z-w)}\gamma V^{\bar{A}}(w)+\cdots, (\bar{A}=1,2,\cdots,N), \\
U_{\mathcal{N}=2}(z)\, V^{\bar{A}}(w) & = -\frac{1}{(z-w)}\gamma V^{\bar{A}}(w)+\cdots, (\bar{A}=N+1,N+2,\cdots,2N), \\
U_{\mathcal{N}=2}(z)\, V^{\bar{A}^*}(w) & = -\frac{1}{(z-w)}\gamma V^{\bar{A}^*}(w)+\cdots, (\bar{A}^*=1^*,2^*,\cdots,N^*), \qquad \text{(C.3)} \\
U_{\mathcal{N}=2}(z)\, V^{\bar{A}^*}(w) & = \frac{1}{(z-w)}\gamma V^{\bar{A}^*}(w)+\cdots, (\bar{A}^*=(N+1)^*,(N+2)^*,\cdots,2N^*).
\end{aligned}
$$

From the explicit charges in (C.3), one can construct any multiple product of spin-1 currents and spin-$\frac{1}{2}$ currents for given $U_{\mathcal{N}=2}$ charge.

Appendix D The complex structures in the extension of large $\mathcal{N}=4$ linear superconformal algebra

One can represent three almost complex structures $h^1_{\tilde{a}\tilde{b}}$, $h^2_{\tilde{a}\tilde{b}}$ and $h^3_{\tilde{a}\tilde{b}}(\equiv h^1_{\tilde{a}\tilde{c}}h^{2\tilde{c}}{}_{\tilde{b}})$ in terms of the $(4N+4)\times(4N+4)$ matrix. We represent $h^i_{\tilde{a}\tilde{b}}$ using the following four block matrices.

$$
h^i_{\tilde{a}\tilde{b}} = \left(\begin{array}{c|c} h^i_{\bar{a}\bar{b}} & h^i_{\bar{a}\hat{b}}=0 \\ \hline h^i_{\hat{a}\bar{b}}=0 & h^i_{\hat{a}\hat{b}} \end{array}\right), \qquad \text{(D.1)}
$$

where the first block is given by $4N\times 4N$ matrix, the second is $4N\times 4$ matrix, the third is $4\times 4N$ matrix, and the last block is given by 4×4 matrix. Note that the coset indices are decomposed into $\tilde{a}=(\bar{a},\hat{a})$ where $\bar{a}=1,2,\cdots,2N,1^*,2^*,\cdots,2N^*$ and $\hat{a}$ runs over the remaining four indices.

The first $4N\times 4N$ block matrices in (D.1) are exactly the same as the complex structures in Wolf space coset as follows:

$$
h^1_{\bar{a}\bar{b}} = \left(\begin{array}{cccc} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{array}\right), \qquad h^2_{\bar{a}\bar{b}} = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right),
$$

$$h^3_{\bar{a}\bar{b}} \equiv h^1_{\bar{a}\bar{c}} h^{2\bar{c}}{}_{\bar{b}} = \left(\begin{array}{cccc} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{array} \right), \tag{D.2}$$

where each entry in the above matrices (D.2) is $N \times N$ matrix. Recall that the coset index $\bar{a}$ in the nonlinear version runs over the $4N$ indices (A.4): the first N indices are given by $1, 2, \cdots, N$, the second N indices are given by $(N+1), (N+2), \cdots, 2N$, third N indices are given by $1^*, 2^*, \cdots, N^*$, and the last N indices are given by $(N+1)^*, (N+2)^*, \cdots, 2N^*$.

Let us represent the last 4×4 block matrices $h^i_{\hat{a}\hat{b}}$ in (D.1) as follows:

$$h^1_{\hat{a}\hat{b}} = \left(\begin{array}{cccc} 0 & 0 & 0 & -\frac{1}{a} \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ \frac{1}{a} & 0 & 0 & 0 \end{array} \right), \qquad h^2_{\hat{a}\hat{b}} = \left(\begin{array}{cccc} 0 & 0 & 0 & \frac{i}{a} \\ 0 & 0 & ia & 0 \\ 0 & -ia & 0 & 0 \\ -\frac{i}{a} & 0 & 0 & 0 \end{array} \right),$$

$$h^3_{\hat{a}\hat{b}} \equiv h^1_{\hat{a}\hat{c}} h^{2\hat{c}}{}_{\hat{b}} = \left(\begin{array}{cccc} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{array} \right), \tag{D.3}$$

where the element a in (D.3) which depends on N is given by

$$a \equiv -\frac{1}{2(N+1)} \left(\sqrt{2N(N+1)} + i\sqrt{2(N+1)(N+2)} \right). \tag{D.4}$$

The $\hat{a}$ indices are given by $(2N+1)$, q, $(2N+1)^*$ and q^* where the index q was defined in (A.5). The N dependences in (D.4) are rather unusual but they can be fixed by the several N cases.

Appendix E The defining OPEs for the higher spin currents in the extension of large $\mathcal{N} = 4$ linear superconformal algebra

The relevant OPEs for the higher spin currents for general N (starting from [27]) are given by

$$\begin{aligned}
\left(\begin{array}{c} \mathbf{G}_{12} \\ \mathbf{G}_{21} \end{array} \right)(z)\, \mathbf{T}^{(1)}(w) &= \frac{1}{(z-w)} \left[\left(\begin{array}{c} -\mathbf{G}_{12} \\ \mathbf{G}_{21} \end{array} \right) + 2\mathbf{T}^{(\frac{3}{2})}_{\mp} \right](w) + \cdots, \\
\left(\begin{array}{c} \mathbf{G}_{11} \\ \mathbf{G}_{22} \end{array} \right)(z)\, \mathbf{T}^{(1)}(w) &= \frac{1}{(z-w)} \left[\left(\begin{array}{c} \mathbf{G}_{11} \\ -\mathbf{G}_{22} \end{array} \right) + 2 \left(\begin{array}{c} \mathbf{U}^{(\frac{3}{2})} \\ \mathbf{V}^{(\frac{3}{2})} \end{array} \right) \right](w) + \cdots, \\
\left(\begin{array}{c} \mathbf{G}_{12} \\ \mathbf{G}_{21} \end{array} \right)(z)\, \mathbf{T}^{(\frac{3}{2})}_{\pm}(w) &= \mp \frac{1}{(z-w)^3} \frac{2(k+1)(N+1)}{(N+k+2)} + \frac{1}{(z-w)^2} \left[\frac{2i(N+1)}{(N+k+2)} \mathbf{A}^{+3} \right.
\end{aligned}$$

$$
\begin{aligned}
& & - & \frac{2i(k+1)}{(N+k+2)}\mathbf{A}^{-3}+\mathbf{T}^{(\mathbf{1})}\Bigg](w) \\
& & + & \frac{1}{(z-w)}\left[\frac{1}{2}\partial\,(\text{pole-2})\mp\mathbf{T}^{(\mathbf{2})}\mp\mathbf{T}\right](w)+\cdots, \\
\left(\begin{array}{c}\mathbf{G}_{12}\\ \mathbf{G}_{21}\end{array}\right)(z)\left(\begin{array}{c}\mathbf{U}^{(\frac{3}{2})}\\ \mathbf{V}^{(\frac{3}{2})}\end{array}\right)(w) & = & - & \frac{1}{(z-w)^2}\frac{2i(N+1)}{(N+k+2)}\mathbf{A}^{+\mp}(w) \\
& & + & \frac{1}{(z-w)}\left[\frac{1}{2}\partial\,(\text{pole-2})+\left(\begin{array}{c}\mathbf{U}_{-}^{(\mathbf{2})}\\ \mathbf{V}_{+}^{(\mathbf{2})}\end{array}\right)\right](w)+\cdots, \\
\left(\begin{array}{c}\mathbf{G}_{12}\\ \mathbf{G}_{21}\end{array}\right)(z)\left(\begin{array}{c}\mathbf{V}^{(\frac{3}{2})}\\ \mathbf{U}^{(\frac{3}{2})}\end{array}\right)(w) & = & - & \frac{1}{(z-w)^2}\frac{2i(k+1)}{(N+k+2)}\mathbf{A}^{-\pm}(w) \\
& & + & \frac{1}{(z-w)}\left[\frac{1}{2}\partial\,(\text{pole-2})+\left(\begin{array}{c}\mathbf{V}_{-}^{(\mathbf{2})}\\ \mathbf{U}_{+}^{(\mathbf{2})}\end{array}\right)\right](w)+\cdots, \\
\left(\begin{array}{c}\mathbf{G}_{11}\\ \mathbf{G}_{22}\end{array}\right)(z)\left(\begin{array}{c}\mathbf{V}^{(\frac{3}{2})}\\ \mathbf{U}^{(\frac{3}{2})}\end{array}\right)(w) & = & \pm & \frac{1}{(z-w)^3}\frac{2(N+1)(k+1)}{(N+k+2)}+\frac{1}{(z-w)^2}\Bigg[-\frac{2i(N+1)}{(N+k+2)}\mathbf{A}^{+3} \\
& & - & \frac{2i(k+1)}{(N+k+2)}\mathbf{A}^{-3}+\mathbf{T}^{(\mathbf{1})}\Bigg](w) \\
& & + & \frac{1}{(z-w)}\left[\frac{1}{2}\partial\,(\text{pole-2})\pm\mathbf{W}^{(\mathbf{2})}\pm\mathbf{T}\right](w)+\cdots, \\
\left(\begin{array}{c}\mathbf{G}_{12}\\ \mathbf{G}_{21}\end{array}\right)(z)\left(\begin{array}{c}\mathbf{V}_{+}^{(\mathbf{2})}\\ \mathbf{U}_{-}^{(\mathbf{2})}\end{array}\right)(w) & = & & \frac{1}{(z-w)^2}\frac{(N+2k+3)}{(N+k+2)}\left[\left(\begin{array}{c}-\mathbf{G}_{22}\\ \mathbf{G}_{11}\end{array}\right)+2\left(\begin{array}{c}\mathbf{V}^{(\frac{3}{2})}\\ \mathbf{U}^{(\frac{3}{2})}\end{array}\right)\right](w) \\
& & + & \frac{1}{(z-w)}\left[\frac{1}{3}\partial\,(\text{pole-2})\mp\left(\begin{array}{c}\mathbf{V}^{(\frac{5}{2})}\\ \mathbf{U}^{(\frac{5}{2})}\end{array}\right)\right](w)+\cdots, \\
\left(\begin{array}{c}\mathbf{G}_{12}\\ \mathbf{G}_{21}\end{array}\right)(z)\,\mathbf{W}^{(\mathbf{2})}(w) & = & & \frac{1}{(z-w)^2}\frac{(k-N)}{2(N+k+2)}\left[\left(\begin{array}{c}\mathbf{G}_{12}\\ \mathbf{G}_{21}\end{array}\right)\mp 2\mathbf{T}_{\mp}^{(\frac{3}{2})}\right](w) \\
& & + & \frac{1}{(z-w)}\left[\frac{1}{3}\partial\,(\text{pole-2})+\mathbf{W}_{\mp}^{(\frac{5}{2})}\right](w)+\cdots, \\
\left(\begin{array}{c}\mathbf{G}_{12}\\ \mathbf{G}_{21}\end{array}\right)(z)\,\mathbf{W}_{\pm}^{(\frac{5}{2})}(w) & = & \mp & \frac{1}{(z-w)^3}\frac{8(k-N)}{3(N+k+2)}\mathbf{T}^{(\mathbf{1})}(w) \\
& & + & \frac{1}{(z-w)^2}\left[\frac{4(k-N)}{3(N+k+2)}\mathbf{T}^{(\mathbf{2})}+4\mathbf{W}^{(\mathbf{2})}\right](w) \\
& & + & \frac{1}{(z-w)}\Bigg[\frac{1}{4}\partial\,(\text{pole-2})\mp\mathbf{W}^{(\mathbf{3})} \\
& & \mp & \frac{4(k-N)}{((4N+5)+(3N+4)k)}\left(\mathbf{T}\,\mathbf{T}^{(\mathbf{1})}-\frac{1}{2}\partial^2\mathbf{T}^{(\mathbf{1})}\right)\Bigg](w)+\cdots,
\end{aligned}
\tag{E.1}
$$

where the notations in [27] have the following relations

$$
A_1\equiv-\mathbf{A}^{+1},\qquad A_2\equiv\mathbf{A}^{+2},\qquad A_3\equiv-\mathbf{A}^{+3},\qquad B_i\equiv\mathbf{A}^{-i},\qquad i=1,2,3.
$$

Note that the structure constants appearing in (E.1) are rather simple form and one can easily figure out the general N dependence for given several N results.

Appendix F The precise relations between the higher spin currents in the linear and nonlinear versions

The exact relations between the 11 currents in the nonlinear and linear versions are given by [36]

$$\begin{aligned}
T(z) &= \mathbf{T}(z) + \frac{1}{(k+N+2)} \left(\mathbf{U}\mathbf{U} + \partial\mathbf{\Gamma}^{\mu}\mathbf{\Gamma}_{\mu}\right)(z), \\
G^{\mu}(z) &= \mathbf{G}^{\mu}(z) \\
&+ \frac{2}{(k+N+2)} \left(\mathbf{U}\mathbf{\Gamma}^{\mu} - \frac{1}{3(k+N+2)}\epsilon_{\mu\nu\rho\sigma}\mathbf{\Gamma}^{\nu}\mathbf{\Gamma}^{\rho}\mathbf{\Gamma}^{\sigma} - 2\alpha^{+i}_{\mu\nu}\mathbf{\Gamma}^{\nu}A^{+i} + 2\alpha^{-i}_{\mu\nu}\mathbf{\Gamma}^{\nu}A^{-i}\right)(z), \\
A^{\pm i}(z) &= \mathbf{A}^{\pm i}(z) - \frac{1}{(k+N+2)}\alpha^{\pm i}_{\mu\nu}\mathbf{\Gamma}^{\mu}\mathbf{\Gamma}^{\nu}(z).
\end{aligned} \tag{F.1}$$

For the higher spin-$\frac{3}{2}$ currents, because of $\mathbf{G}'^{\mu}(z) = G'^{\mu}(z)$ in (4.18), the differences between the higher spin-$\frac{3}{2}$ currents in the nonlinear and linear versions (4.19) come from the differences between the spin-$\frac{3}{2}$ current G^{μ} and the spin-$\frac{3}{2}$ current $\mathbf{G}^{\mu}$ in (F.1). See the equation (4.19) (and (3.21) and (3.22) of [2]).

For the higher spin-2 currents, one has

$$\begin{aligned}
U^{(2)}_{+}(z) &= \mathbf{U}^{(\mathbf{2})}_{+}(z) + \frac{1}{(N+k+2)}\left[\mathbf{\Gamma}_{11}\mathbf{G}'_{21} - \mathbf{\Gamma}_{21}\mathbf{G}'_{11} + 2A^{+3}A^{--}\right](z), \\
U^{(2)}_{-}(z) &= \mathbf{U}^{(\mathbf{2})}_{-}(z) + \frac{1}{(N+k+2)}\left[\mathbf{\Gamma}_{12}\mathbf{G}'_{11} - \mathbf{\Gamma}_{11}\mathbf{G}'_{12} + 2A^{+-}A^{-3}\right](z), \\
V^{(2)}_{+}(z) &= \mathbf{V}^{(\mathbf{2})}_{+}(z) + \frac{1}{(N+k+2)}\left[\mathbf{\Gamma}_{21}\mathbf{G}'_{22} - \mathbf{\Gamma}_{22}\mathbf{G}'_{21} - 2A^{++}A^{-3}\right](z), \\
V^{(2)}_{-}(z) &= \mathbf{V}^{(\mathbf{2})}_{-}(z) + \frac{1}{(N+k+2)}\left[\mathbf{\Gamma}_{22}\mathbf{G}'_{12} - \mathbf{\Gamma}_{12}\mathbf{G}'_{22} - 2A^{+3}A^{-+}\right](z), \\
T^{(2)}(z) &= \mathbf{T}^{(\mathbf{2})}(z) + \frac{1}{(N+k+2)}\left[\mathbf{\Gamma}_{22}\mathbf{G}'_{11} - \mathbf{\Gamma}_{11}\mathbf{G}'_{22} + A^{+i}A^{+i} + A^{-i}A^{-i} + 2A^{+3}A^{-3}\right](z) \\
&+ \frac{(N+k)}{(2Nk+N+k)}T(z), \\
W^{(2)}(z) &= \mathbf{W}^{(\mathbf{2})}(z) + \frac{1}{(N+k+2)}\left[\mathbf{\Gamma}_{12}\mathbf{G}'_{21} - \mathbf{\Gamma}_{21}\mathbf{G}'_{12} + A^{+i}A^{+i} + A^{-i}A^{-i} - 2A^{+3}A^{-3}\right](z) \\
&+ T(z).
\end{aligned} \tag{F.2}$$

In (F.2), the N dependence can be determined without any difficulty.

For the higher spin-$\frac{5}{2}$ currents, one obtains

$$
\begin{aligned}
U^{(\frac{5}{2})}(z) &= \mathbf{U}^{(\frac{5}{2})}(z) + \frac{1}{(N+k+2)} \Big[-2\partial \mathbf{\Gamma_{11}} \mathbf{T^{(1)}} + \mathbf{\Gamma_{11}} \partial \mathbf{T^{(1)}} + 2\mathbf{\Gamma_{11}} \mathbf{W^{(2)}} - 2\mathbf{\Gamma_{12}} \mathbf{U_{+}^{(2)}} - 2\mathbf{\Gamma_{21}} \mathbf{U_{-}^{(2)}} \\
&+ i\mathbf{A}^{+-}\mathbf{G}'_{21} + i\mathbf{A}^{+3}\mathbf{G}'_{11} - i\mathbf{A}^{--}\mathbf{G}'_{12} - i\mathbf{A}^{-3}\mathbf{G}'_{11} + \mathbf{U}\mathbf{G}'_{11} - iA^{+-}G'_{21} + 2iA^{--}G'_{12} \\
&+ 2iA^{-3}G'_{11} - iA^{+-}G_{21} - 2iA^{-3}G_{11} + \frac{2}{3}\partial G'_{11} - \frac{2}{3}\partial G_{11} \Big](z), \\
V^{(\frac{5}{2})}(z) &= \mathbf{V}^{(\frac{5}{2})}(z) + \frac{1}{(N+k+2)} \Big[2\partial \mathbf{\Gamma_{22}} \mathbf{T^{(1)}} - \mathbf{\Gamma_{22}} \partial \mathbf{T^{(1)}} + 2\mathbf{\Gamma_{12}} \mathbf{V_{+}^{(2)}} + 2\mathbf{\Gamma_{21}} \mathbf{V_{-}^{(2)}} + 2\mathbf{\Gamma_{22}} \mathbf{W^{(2)}} \\
&+ i\mathbf{A}^{++}\mathbf{G}'_{12} + i\mathbf{A}^{+3}\mathbf{G}'_{22} - i\mathbf{A}^{-+}\mathbf{G}'_{21} - i\mathbf{A}^{-3}\mathbf{G}'_{22} - \mathbf{U}\mathbf{G}'_{22} - 2iA^{++}G'_{12} - 2iA^{+3}G'_{22} \\
&+ iA^{-+}G'_{21} - 2iA^{+3}G_{22} - iA^{-+}G_{21} + \frac{2}{3}\partial G'_{22} + \frac{2}{3}\partial G_{22} \Big](z), \qquad \text{(F.3)} \\
W_{+}^{(\frac{5}{2})}(z) &= \mathbf{W}_{+}^{(\frac{5}{2})}(z) + \frac{1}{(N+k+2)} \Big[-2\partial \mathbf{\Gamma_{21}} \mathbf{T^{(1)}} + \mathbf{\Gamma_{21}} \partial \mathbf{T^{(1)}} + 2\mathbf{\Gamma_{21}} \mathbf{T^{(2)}} + 2\mathbf{\Gamma_{11}} \mathbf{V_{+}^{(2)}} + 2\mathbf{\Gamma_{22}} \mathbf{U_{+}^{(2)}} \\
&+ i\mathbf{A}^{++}\mathbf{G}'_{11} - i\mathbf{A}^{+3}\mathbf{G}'_{21} + i\mathbf{A}^{--}\mathbf{G}'_{22} - i\mathbf{A}^{-3}\mathbf{G}'_{21} + \mathbf{U}\mathbf{G}'_{21} - iA^{++}G'_{11} + 2iA^{+3}G'_{21} \\
&- iA^{--}G'_{22} + 2iA^{-3}G'_{21} - iA^{++}G_{11} + 2iA^{+3}G_{21} + iA^{--}G_{22} - 2iA^{-3}G_{21} - \frac{4}{3}\partial G_{21} \Big](z), \\
W_{-}^{(\frac{5}{2})}(z) &= \mathbf{W}_{-}^{(\frac{5}{2})}(z) + \frac{1}{(N+k+2)} \Big[2\partial \mathbf{\Gamma_{12}} \mathbf{T^{(1)}} - \mathbf{\Gamma_{12}} \partial \mathbf{T^{(1)}} + 2\mathbf{\Gamma_{12}} \mathbf{T^{(2)}} - 2\mathbf{\Gamma_{11}} \mathbf{V_{-}^{(2)}} - 2\mathbf{\Gamma_{22}} \mathbf{U_{-}^{(2)}} \\
&+ i\mathbf{A}^{+-}\mathbf{G}'_{22} - i\mathbf{A}^{+3}\mathbf{G}'_{12} + i\mathbf{A}^{-+}\mathbf{G}'_{11} - i\mathbf{A}^{-3}\mathbf{G}'_{12} - \mathbf{U}\mathbf{G}'_{12} - iA^{+-}G'_{22} + 2iA^{+3}G'_{12} \\
&- iA^{-+}G'_{11} + 2iA^{-3}G'_{12} + iA^{+-}G_{22} - 2iA^{+3}G_{12} - iA^{-+}G_{11} + 2iA^{-3}G_{12} - \frac{4}{3}\partial G_{12} \Big](z).
\end{aligned}
$$

In this case, the expression (F.3) reveals the N dependence clearly.

Finally, for the higher spin-3 current, the following relation holds

$$
\begin{aligned}
W^{(3)}(z) &= \Big[\mathbf{W^{(3)}} + a_1 \mathbf{T^{(1)}} \mathbf{T} \\
&+ \frac{1}{(N+k+2)} \left(\sum_{\mu=0}^{3} (\mathbf{\Gamma}^{\mu} \partial \mathbf{G}'^{\mu} - 3\partial \mathbf{\Gamma}^{\mu} \mathbf{G}'^{\mu}) + 2(\partial \mathbf{U} \mathbf{T^{(1)}} - \mathbf{U} \partial \mathbf{T^{(1)}}) \right) \Big](z) \\
&+ a_2 T^{(1)} T(z) - \frac{1}{(N+k+2)} \Big[2G_{12}G_{21} - 2iA^{++}U_{-}^{(2)} - 2iA^{--}V_{-}^{(2)} - 4iA^{+3}T^{(2)} \\
&+ a_4 A^{-3} T - 4iA^{-3}T^{(2)} + iA^{+3}\partial T^{(1)} + iA^{-3}\partial T^{(1)} - iT^{(1)}\partial A^{+3} - iT^{(1)}\partial A^{-3} - \partial W^{(2)} \\
&- \partial T \Big](z) - \frac{1}{(N+k+2)^2} \Big[8iA^{+3}A^{+-}A^{++} + 8iA^{-3}A^{+-}A^{++} + 8iA^{+3}A^{+3}A^{+3} \\
&+ 16iA^{+3}A^{+3}A^{-3} + 8iA^{+3}A^{-3}A^{-3} + a_5 \partial A^{+-}A^{++} + a_6 A^{+-}\partial A^{++} \\
&- (k+1)A^{-+}\partial A^{--} + 6A^{+3}\partial A^{+3} + a_7 A^{+3}\partial A^{-3} + a_8 A^{-3}\partial A^{+3} - 2A^{-3}\partial A^{-3} \\
&+ a_9 \partial^2 A^{+3} - \frac{8ik}{3}\partial^2 A^{-3} + a_3 A^{+3} T + (k-1)A^{--}\partial A^{-+} \Big](z), \qquad \text{(F.4)}
\end{aligned}
$$

where the coefficients in (F.4) are given by

$$
\begin{aligned}
a_1 &= \frac{4(k-N)}{(4N+5+(3N+4)k)}, \\
a_2 &= \frac{8(N-k)}{(5N+4+(6N+5)k)}, \\
a_3 &= \frac{8i(N+k+2)(4N^2+4N+(6N^2+13N+4)k+(8N+5)k^2)}{(2Nk+N+k)(5N+4+(6N+5)k)}, \\
a_4 &= \frac{8i(-2N^2-3N+k)k}{(2Nk+N+k)(5N+4+(6N+5)k)}, \\
a_5 &= (N-1), \\
a_6 &= -(N+1), \\
a_7 &= 2(k-N-1), \\
a_8 &= -2(k-N-3), \\
a_9 &= \frac{i(5N+3)}{3}.
\end{aligned}
\tag{F.5}
$$

In this case, the general N behavior can be obtained by taking some power of N in the fractional expressions of the level k from the several N results. It turns out that the quadratic in N occurs. See, for example, the numerator of the coefficient a_3 in (F.5).

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